MATH 436: Solutions for Problem Set 4

3.3.12: (a) The 3×3 system of pdes is given by

$$\mathbf{u}_y + A\mathbf{u}_x = C\mathbf{u} \text{ where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -2 & 1 & 5 \\ 0 & 3 & 7 \\ 1 & -3 & -10 \end{bmatrix}.$$

We have to reduce this to normal form. The first thing we need to calculate is ω where

$$|I - \omega A| = 0 \Longleftrightarrow \left|A - \omega^{-1} I\right| = \begin{bmatrix} 1-\omega^{-1} & 0 & 1 \\ 0 & 2-\omega^{-1} & 3 \\ 0 & 0 & -1-\omega^{-1} \end{bmatrix} = 0$$

$$\Longrightarrow \left(1-\omega^{-1}\right)\left(1+\omega^{-1}\right)\left(2-\omega^{-1}\right) = 0 \Longrightarrow \omega = -1,\ 1/2 \text{ and } 1.$$

Since the ω are real and distinct the system is totally hyperbolic. The characteristic curves are the level surfaces determined by

$$\frac{dy}{dx} = \omega,$$

which implies that they are the straight lines $\xi = y+x$, $\eta = y-x/2$ and $\zeta = y-x$ for constant ξ, η and ζ.
(b) To reduce to normal form we need to compute the right eigenvectors associated with each eigenvalue. For $\omega = -1$, we have

$$\left(A - \omega^{-1} I\right)\mathbf{r}_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{r}_1 = 0 \Longrightarrow \mathbf{r}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

For $\omega = 1/2$, we have

$$\left(A - \omega^{-1} I\right)\mathbf{r}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix}\mathbf{r}_2 = 0 \Longrightarrow \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\omega = 1$, we have

$$\left(A - \omega^{-1} I\right)\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{r}_3 = 0 \Longrightarrow \mathbf{r}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, introducing the matrix R whose columns are the eigenvectors, we have

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \Longrightarrow R^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Thus, introducing the transformation of dependent variables defined by

$$\mathbf{u} = R\mathbf{v},$$

it follows from the pde that

$$(R\mathbf{v})_y + A(R\mathbf{v})_x = CR\mathbf{v} \Longrightarrow \mathbf{v}_y + R^{-1}AR\mathbf{v}_x = R^{-1}CR\mathbf{v},$$

since R's elements are all constants. And since

$$R^{-1}AR = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$R^{-1}CR = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 5 \\ 0 & 3 & 7 \\ 1 & -3 & -10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & -\frac{5}{2} \\ 1 & 0 & 7 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{15}{2} \end{bmatrix},$$

the normal form for the pde is

$$\mathbf{v}_y + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{v}_x = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & -\frac{5}{2} \\ 1 & 0 & 7 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{15}{2} \end{bmatrix} \mathbf{v}.$$

3.3.19: (a) The 2×2 system of pdes is given by

$$\mathbf{u}_y + A\mathbf{u}_x = \mathbf{0} \text{ where } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}. \tag{1}$$

If $y = h(x)$ are the characteristics, then $h(x)$ must satisfy

$$|I - h'(x)A| = \begin{vmatrix} 1-h' & -2h' \\ -2h' & 1-h' \end{vmatrix} = (1-h')^2 - 4(h')^2 = 0$$

$$\Longrightarrow 1 - h' = \pm 2h' \Longrightarrow h' = \frac{1}{3} \text{ or } -1,$$

which implies that the characteristic curves are given by $y = x/3 + C$ and $y = -x + \widetilde{C}$, respectively. The specific characteristics given in the question have $\widetilde{C} = C = 0$. Note that $1/h'$ is an "eigenvalue" of A.
(b) Suppose that $\mathbf{u}(x, h(x)) = \mathbf{f}(x)$, i.e., $\mathbf{u}$ is specified on a characteristic curve, so that this is potentially a *characteristic initial value problem.* It follows that

$$\frac{d\mathbf{u}(x, h(x))}{dx} = \mathbf{u}_x + h'\mathbf{u}_y = \mathbf{f}' \Longrightarrow \mathbf{u}_x = -h'\mathbf{u}_y + \mathbf{f}',$$

which when substituted into (1) leads to

$$(I - h'A)\mathbf{u}_y = -A\mathbf{f}', \tag{2}$$

which is just (3.3.36) in the textbook (for the particular pde in this question). Since the characteristic curves (i.e., $y = h(x)$) have been chosen by demanding that $\det(I - h'A) = 0$, then if we let $\mathbf{l}$ be a *left eigenvector* associated with the eigenvalue $1/h'$, i.e., $\mathbf{l}\cdot (I - h'A) = \mathbf{0}$, it follows from (2) that

$$\mathbf{l}(I - h'A)\,\mathbf{u}_y = 0 = -\mathbf{l}A\mathbf{f}' \Longrightarrow \mathbf{l}A\mathbf{f}' = 0,$$

which is a compatibility condition on the allowed initial data $\mathbf{f}(x)$ on the characteristic.

In component form, (2) is given by

$$\begin{bmatrix} 1-h' & -2h' \\ -2h' & 1-h' \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_y = -\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_x, \tag{3}$$

where we have written $\mathbf{u} = (u_1,\, u_2)^\top$ and $\mathbf{f} = (f_1,\, f_2)^\top$.
For $h' = 1/3$, (3) reduces to

$$\begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_y = -\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_x, \tag{4}$$

from which we immediately see that

$$(1,\, 1)\cdot \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \mathbf{0},$$

i.e., $\mathbf{l} = (1,\, 1)^\top$, so that it follows from (4) that

$$(1,\, 1)\cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_x = 3\,(f_1 + f_2)_x = 0 \Longrightarrow f_2(x) = -f_1(x) + C,$$

where C is an arbitrary constant.
For $h' = -1$, (3) reduces to

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_y = -\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_x, \tag{5}$$

from which we immediately see that

$$(1,\, -1)\cdot \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \mathbf{0},$$

i.e., $\mathbf{l} = (1, -1)^\top$, so that it follows from (5) that

$$(1,\, -1)\cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}_x = (-f_1 + f_2)_x = 0 \Longrightarrow f_2(x) = f_1(x) + C,$$

where C is an arbitrary constant.
(c) Perhaps the most direct way to get the general solution is to first reduce the system (1) to normal form. The *right eigenvectors* of A associated with the

eigenvalues $1/h'$, i.e., -1 and 3, and denoted by $\mathbf{r}_{-1}$ and $\mathbf{r}_3$, respectively, are determined by

$$(I+A)\,\mathbf{r}_{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{r}_{-1} = \mathbf{0} \Longrightarrow \mathbf{r}_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\left(I - \tfrac{1}{3}A\right) \mathbf{r}_3 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \mathbf{r}_3 = \mathbf{0} \Longrightarrow \mathbf{r}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that $\mathbf{r}_{-1} \cdot \mathbf{r}_3 = 0$. Thus, introducing the transformation

$$\mathbf{u} \equiv R\mathbf{v} = R\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \text{ where } R \equiv \begin{bmatrix} \mathbf{r}_{-1} & \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Longrightarrow R^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

into (1), leads to

$$\mathbf{v}_y + D\mathbf{v}_x = \mathbf{0}, \text{ where } D \equiv R^{-1}AR = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix},$$

or, in component form,

$$\partial_y v_1 - \partial_x v_1 = 0 \text{ and } \partial_y v_2 + 3\partial_x v_2 = 0.$$

Therefore, using the method of characteristics, the general solution for $\mathbf{v}$ is given by

$$\mathbf{v} = \begin{pmatrix} g\,(x+y) \\ h\,(x-3y) \end{pmatrix},$$

where $g\,(x+y)$ and $h\,(x-3y)$ are arbitrary functions of their arguments. Thus, the general solution to (1) is given

$$\mathbf{u} = R\mathbf{v} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} g\,(x+y) \\ h\,(x-3y) \end{pmatrix} = \begin{pmatrix} h\,(x-3y) + g\,(x+y) \\ h\,(x-3y) - g\,(x+y) \end{pmatrix}.$$

Hence, for the problem in which the initial data is specified along the characteristic $y = x/3$, i.e.,

$$\mathbf{u}\,(x, x/3) = \begin{pmatrix} f\,(x) \\ -f\,(x) + C \end{pmatrix} = R\begin{pmatrix} g\,(4x/3) \\ h\,(0) \end{pmatrix},$$

(as determined in Part (b)), it follows that

$$\begin{pmatrix} g\,(x) \\ h\,(0) \end{pmatrix} = R^{-1}\begin{pmatrix} f\,(3x/4) \\ -f\,(3/4x) + C \end{pmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} f\,(3x/4) \\ -f\,(3/4x) + C \end{pmatrix} = \begin{pmatrix} f\,(3x/4) - C/2 \\ C/2 \end{pmatrix},$$

from which we conclude

$$C = 2h(0),$$

$$g(x) = f(3x/4) - h(0).$$

It therefore follows that

$$\mathbf{u}(x,y) = \begin{pmatrix} h(x-3y) - h(0) + f(3(x+y)/4) \\ h(x-3y) + h(0) - f(3(x+y)/4) \end{pmatrix},$$

where $h(x-3y)$ is arbitrary, i.e., the solution is not unique.
Similarly, for the problem in which the initial data is specified along the characteristic $y = -x$, i.e.,

$$\mathbf{u}(x,-x) = \begin{pmatrix} f(x) \\ f(x) + C \end{pmatrix} = R \begin{pmatrix} g(0) \\ h(4x) \end{pmatrix},$$

(as determined in Part (b)), that

$$\begin{pmatrix} g(0) \\ h(x) \end{pmatrix} = R^{-1} \begin{pmatrix} f(x/4) \\ f(x/4) + C \end{pmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} f(x/4) \\ f(x/4) + C \end{pmatrix} = \begin{pmatrix} -C/2 \\ f(x/4) + C/2 \end{pmatrix},$$

from which we conclude that

$$C = -2g(0),$$

$$h(x) = f(x/4) - g(0).$$

It therefore follows that

$$\mathbf{u}(x,y) = \begin{pmatrix} f((x-3y)/4) + g(x+y) - g(0) \\ f((x-3y)/4) - g(x+y) - g(0) \end{pmatrix},$$

where $g(x+y)$ is arbitrary, i.e., the solution is not unique.

3.3.20: Euler's equations for isentropic (i.e., constant entropy) flow are given by

$$u_t + u\,u_x + \left(c^2/\rho\right)\rho_x = 0,$$

$$\rho_t + u\rho_x + \rho\,u_x = 0,$$

where u and ρ are the velocity and density, respectively. This pair of equations may be written as the 2×2 system

$$\begin{bmatrix} u & c^2/\rho \\ \rho & u \end{bmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_x + \begin{pmatrix} u \\ \rho \end{pmatrix}_t = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which we have written in the form of (3.3.73) in the textbook.

(a) To reduce these equations to characteristic normal form we need to solve the eigenvalue problem

$$\begin{bmatrix} u & c^2/\rho \\ \rho & u \end{bmatrix}^\top \mathbf{r} = \lambda \mathbf{r} \Longrightarrow \begin{vmatrix} u-\lambda & \rho \\ c^2/\rho & u-\lambda \end{vmatrix} = 0 \Longrightarrow \lambda = u \pm c.$$

The corresponding right eigenvectors are determined as follows. For $\lambda = u + c$, we have

$$\begin{bmatrix} -c & \rho \\ c^2/\rho & -c \end{bmatrix} \mathbf{r}_+ = \mathbf{0} \Longrightarrow \mathbf{r}_+ = \begin{pmatrix} \rho \\ c \end{pmatrix}.$$

For $\lambda = u - c$, we have

$$\begin{bmatrix} c & \rho \\ c^2/\rho & c \end{bmatrix} \mathbf{r}_- = \mathbf{0} \Longrightarrow \mathbf{r}_- = \begin{pmatrix} \rho \\ -c \end{pmatrix}.$$

Now, we left-multiply the original system by the transposed right eigenvectors. For $\mathbf{r}_+$, we have

$$\begin{pmatrix} \rho & c \end{pmatrix} \begin{bmatrix} u & c^2/\rho \\ \rho & u \end{bmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_x + \begin{pmatrix} \rho & c \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_t = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} \rho & c \end{pmatrix} \left[\begin{pmatrix} u \\ \rho \end{pmatrix}_t + (u+c) \begin{pmatrix} u \\ \rho \end{pmatrix}_x \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and thus to

$$\rho \left[u_t + (u+c)\, u_x \right] + c \left[\rho_t + (u+c)\, \rho_x \right] = 0, \tag{1}$$

from which we conclude that

$$\rho \frac{du}{dt} + c \frac{d\rho}{dt} = 0 \text{ on } \frac{dx}{dt} = u + c. \tag{2}$$

For $\mathbf{r}_-$, we have

$$\begin{pmatrix} \rho & -c \end{pmatrix} \begin{bmatrix} u & c^2/\rho \\ \rho & u \end{bmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_x + \begin{pmatrix} \rho & -c \end{pmatrix} \begin{pmatrix} u \\ \rho \end{pmatrix}_t = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} \rho & -c \end{pmatrix} \left[\begin{pmatrix} u \\ \rho \end{pmatrix}_t + (u-c) \begin{pmatrix} u \\ \rho \end{pmatrix}_x \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and thus to

$$\rho \left[u_t + (u-c)\, u_x \right] - c \left[\rho_t + (u-c)\, \rho_x \right], \tag{3}$$

from we conclude that

$$\rho \frac{du}{dt} - c \frac{d\rho}{dt} = 0 \text{ on } \frac{dx}{dt} = u - c. \tag{4}$$

Equations (1) and (3) or equations (2) and (4) are, equivalently, the characteristic normal form for isentropic flow. Equations (2) and (4) may be summarized in the form

$$\rho \frac{du}{dt} \pm c \frac{d\rho}{dt} = 0 \text{ on } \frac{dx}{dt} = u \pm c.$$

Dividing through by $\rho > 0$ and assuming c is constant allows us to write

$$\frac{d}{dt}\left(u \pm c \ln \rho\right) = 0 \text{ on } \frac{dx}{dt} = u \pm c,$$

which can be integrated to yield the Riemann invariants

$$u \pm c \ln \rho = \text{constant, on } \frac{dx}{dt} = u \pm c$$

3.4.7: The backward heat equation is

$$u_t = -\rho u_{xx},\ 0 < x < \pi,\ t > 0.$$

Let us consider the solution

$$u\left(x,t\right) = \frac{\exp\left(\rho n^2 t\right) \sin\left(nx\right)}{n},\ n > 0.$$

Direct substitution verifies that this is a solution. Clearly,

$$u\left(x,0\right) = \frac{\sin\left(nx\right)}{n} \Longrightarrow \left|u\left(x,0\right)\right| \leq \frac{1}{n},$$

can be made as "small" as we want by allowing n to be arbitrarily large. However, *for all* $t > 0$, $u\left(x,t\right)$ will, generically, be exponentially large when n is arbitrarily large. Thus "small" initial data does not necessarily lead to "small" solutions. This violates Hadamard's third condition. So the pde is not well-posed.

3.5.4: The pde (1.1.15) is given by

$$v_t + c v_x = r v_{xx} \text{ (where } r \equiv D/2 > 0\text{).} \tag{1}$$

We must compute the stability index. Assuming a normal mode solution in the form

$$v = a \exp\left(ikx + \lambda t\right) + c.c.,$$

where $k \in \mathbb{R}$ (the wave number), $\lambda \in \mathbb{C}$ (the complex-valued "growth rate") and a is the amplitude coefficient, leads to the relation

$$\lambda + ick = -rk^2 \Longrightarrow \lambda = -ick - rk^2.$$

Thus,

$$\mathrm{Re}\left[\lambda\left(k\right)\right] = -rk^2 \text{ and } \Omega = lub\left\{\mathrm{Re}\left[\lambda\left(k\right)\right]\right\} = lub\left\{-rk^2\right\} = 0.$$

Since $\mathrm{Re}\left[\lambda\left(k\right)\right] \leq 0$ for all $k \in \mathbb{R}$ and $\mathrm{Re}\left[\lambda\left(k\right)\right] = 0$ only for $k = 0$, it follows that the pde (1) is dissipative.