MATH 436: Solutions for Problem Set 3

3.1.5: The pde is given by

$$u_{xx}+2u_{xy}+u_{yy}+5u_x+3u_y+u=0.$$

The $\omega(x,y)$ functions are determined by

$$\omega^2+2\omega+1=0 \Longleftrightarrow (\omega+1)^2=0 \Longleftrightarrow \omega=-1.$$

Since $\omega=-1$ is double root, the pde is *parabolic*.

To transform the pde into canonical form we first introduce the characteristic variable ξ, as determined from

$$\left(\frac{dy}{dx}\right)_\xi=-\omega=1 \Longrightarrow \xi=y-x,$$

and introduce the additional (independent) variable, say, $\eta=y+x$. Thus, introducing the coordinate transformation $(x,y)\to(\xi,\eta)$ leads to

$$\partial_x=\xi_x\partial_\xi+\eta_x\partial_\eta=-\partial_\xi+\partial_\eta \Longrightarrow \partial_{xx}=\partial_{\xi\xi}-2\partial_{\xi\eta}+\partial_{\eta\eta},$$
$$\partial_y=\xi_y\partial_\xi+\eta_y\partial_\eta=\partial_\xi+\partial_\eta \Longrightarrow \partial_{yy}=\partial_{\xi\xi}+2\partial_{\xi\eta}+\partial_{\eta\eta},$$
$$\partial_{xy}=-\partial_{\xi\xi}+\partial_{\eta\eta},$$

which, when substituted into the pde, yields

$$\left(\partial_{\xi\xi}-2\partial_{\xi\eta}+\partial_{\eta\eta}\right)u+2\left(-\partial_{\xi\xi}+\partial_{\eta\eta}\right)u+\left(\partial_{\xi\xi}+2\partial_{\xi\eta}+\partial_{\eta\eta}\right)u$$
$$+5\left(-\partial_\xi+\partial_\eta\right)u+3\left(\partial_\xi+\partial_\eta\right)+u=0,$$

which simplifies to

$$u_{\eta\eta}+2u_\eta-\frac{1}{2}u_\xi+\frac{1}{4}u=0. \qquad (1)$$

The u and u_η terms can be eliminated by introducing the dependent variable transformation

$$u(\xi,\eta)=\exp(\alpha\xi+\beta\eta)\,v(\xi,\eta), \qquad (2)$$

where α and β are yet-to-be-determined constants. It follows from (2) that

$$u_\xi=(\alpha v+v_\xi)\exp(\alpha\xi+\beta\eta),\; u_\eta=(\beta v+v_\eta)\exp(\alpha\xi+\beta\eta),$$
$$u_{\eta\eta}=\left(\beta^2v+2\beta v_\eta+v_{\eta\eta}\right)\exp(\alpha\xi+\beta\eta),$$

which when substituted into (1), yields

$$v_{\eta\eta}+2(\beta+1)v_\eta+\left(\beta^2+2\beta-\frac{\alpha}{2}+\frac{1}{4}\right)v-\frac{1}{2}v_\xi=0. \qquad (3)$$

Thus, if we set

$$\beta+1=\beta^2+2\beta-\frac{\alpha}{2}+\frac{1}{4}=0 \Longleftrightarrow \beta=-1 \text{ and } \alpha=-\frac{3}{2},$$

it follows that (3) reduces to

$$v_\xi = 2v_{\eta\eta}.$$

3.1.6: The pde is given by

$$u_{xx} - 6u_{xy} + 12u_{yy} + 4u_x - u = \sin(xy).$$

The $\omega(x,y)$ functions are determined by

$$\omega^2 - 6\omega + 12 = 0 \Longleftrightarrow \omega^\pm = 3 \pm i\sqrt{3}.$$

Since the solutions for ω are complex, the pde is *elliptic*.

To transform the pde into canonical form we first introduce the characteristic variables ξ and η, as determined by

$$\left(\frac{dy}{dx}\right)_\xi = -\omega^+ = -3 - i\sqrt{3} \Longrightarrow \xi = y + \left(3 + i\sqrt{3}\right)x,$$

$$\left(\frac{dy}{dx}\right)_\eta = -\omega^- = -3 + i\sqrt{3} \Longrightarrow \xi = y + \left(3 - i\sqrt{3}\right)x,$$

and then, finally, define the variables α and β given by

$$\alpha = \frac{\xi + \eta}{2} = y + 3x \text{ and } \beta = \frac{\xi - \eta}{2i} = \sqrt{3}x.$$

Thus, introducing the coordinate transformation $(x,y) \to (\alpha,\beta)$ leads to

$$\partial_x = 3\partial_\alpha + \sqrt{3}\partial_\beta \Longrightarrow \partial_{xx} = 9\partial_{\alpha\alpha} + 6\sqrt{3}\partial_{\alpha\beta} + 3\partial_{\beta\beta},$$

$$\partial_y = \partial_\alpha \Longrightarrow \partial_{yy} = \partial_{\alpha\alpha},\ \partial_{xy} = 3\partial_{\alpha\alpha} + \sqrt{3}\partial_{\alpha\beta},$$

which, when substituted into the pde, yields

$$\left(9\partial_{\alpha\alpha} + 6\sqrt{3}\partial_{\alpha\beta} + 3\partial_{\beta\beta}\right)u - 6\left(3\partial_{\alpha\alpha} + \sqrt{3}\partial_{\alpha\beta}\right)u + 12u_{\alpha\alpha}$$

$$+4\left(3\partial_\alpha + \sqrt{3}\partial_\beta\right)u - u = \sin\left[\beta\left(\alpha - \sqrt{3}\beta\right)/\sqrt{3}\right],$$

which simplifies to

$$u_{\alpha\alpha} + u_{\beta\beta} + 4u_\alpha + \tfrac{4}{\sqrt{3}}u_\beta - \tfrac{1}{3}u = \tfrac{1}{3}\sin\left[\beta\left(\alpha - \sqrt{3}\beta\right)/\sqrt{3}\right]. \quad (1)$$

The u_α and u_β terms in (1) can be eliminated by introducing the dependent variable transformation

$$u(\alpha,\beta) = \exp(\lambda\alpha + \gamma\beta)\, v(\alpha,\beta), \quad (2)$$

where λ and γ are yet-to-be-determined constants. It follows from (2) that

$$u_\alpha = (\lambda v + v_\alpha)\exp(\lambda\alpha + \gamma\beta),\ u_\beta = (\gamma v + v_\beta)\exp(\lambda\alpha + \gamma\beta),$$

$$u_{\alpha\alpha} = \left(\lambda^2 v + 2\lambda v_\alpha + v_{\alpha\alpha}\right) \exp\left(\lambda\alpha + \gamma\beta\right),$$
$$u_{\beta\beta} = \left(\gamma^2 v + 2\gamma v_\beta + v_{\beta\beta}\right) \exp\left(\lambda\alpha + \gamma\beta\right),$$

which when substituted into (1), yields

$$v_{\alpha\alpha} + v_{\beta\beta} + (2\lambda + 4)\, v_\alpha + \left(2\gamma + 4/\sqrt{3}\right) v_\beta + \left(\lambda^2 + \gamma^2 + 4\lambda + 4\gamma/\sqrt{3} - \frac{1}{3}\right) v$$
$$= \tfrac{1}{3} \sin\left[\beta\left(\alpha - \sqrt{3}\beta\right)/\sqrt{3}\right] \exp\left[-\left(\lambda\alpha + \gamma\beta\right)\right]. \qquad (3)$$

Thus, if we set

$$2\lambda + 4 = 2\gamma + 4/\sqrt{3} = 0 \Longleftrightarrow \lambda = -2 \text{ and } \gamma = -2/\sqrt{3},$$

it follows that (3) reduces to

$$v_{\alpha\alpha} + v_{\beta\beta} - \tfrac{17}{3} v = \tfrac{1}{3} \sin\left[\beta\left(\alpha - \sqrt{3}\beta\right)/\sqrt{3}\right] \exp\left[2\alpha + 2\beta/\sqrt{3}\right].$$

3.1.7: Note to the student: *Tricomi's equation* is perhaps the simplest model describing irrotational transonic aerodynamics. You can find a derivation in, for example, the book *Transonic Aerodynamics* (North-Holland Series in Applied Mathematics and Mechanics, Volume 30, 1986) by J. D. Cole and L. P. Cook.
(a) Tricomi's pde is

$$u_{xx} + x u_{yy} = 0. \qquad (1)$$

The $\omega^\pm$ roots are given by

$$\omega^\pm = \pm\frac{\sqrt{-4x}}{2} \Longrightarrow \omega^- = -\sqrt{-x} \text{ and } \omega^+ = \sqrt{-x}.$$

The pde is hyperbolic, elliptic and parabolic for $x < 0$, $x > 0$ and along $x = 0$, respectively.
(b) In the hyperbolic region, the characteristic variables are determined by

$$\left(\frac{dy}{dx}\right)_\xi = -\omega^+ = -\sqrt{-x} \Longrightarrow \xi = y - \tfrac{2}{3}\left(-x\right)^{\frac{3}{2}},$$

$$\left(\frac{dy}{dx}\right)_\eta = -\omega^- = \sqrt{-x} \Longrightarrow \eta = y + \tfrac{2}{3}\left(-x\right)^{\frac{3}{2}},$$

with the "inverse" relations

$$x = -\left[\frac{3\left(\eta - \xi\right)}{4}\right]^{\frac{2}{3}} \text{ and } y = \frac{\xi + \eta}{2}.$$

The derivatives will transform according to

$$u_x = u_\xi \xi_x + u_\eta \eta_x = \sqrt{-x}\left(\partial_\xi - \partial_\eta\right) u,$$

$$
\begin{aligned}
u_{xx} &= \left[\sqrt{-x}\left(\partial_\xi - \partial_\eta\right) u\right]_x \\
&= -x\left(\partial_\xi - \partial_\eta\right)^2 u - \frac{1}{2\sqrt{-x}}\left(\partial_\xi - \partial_\eta\right) u \\
&= -x\left(\partial_{\xi\xi} - 2\partial_{\xi\eta} + \partial_{\eta\eta}\right) u - \frac{1}{2\sqrt{-x}}\left(\partial_\xi - \partial_\eta\right) u,
\end{aligned}
$$

where care has been taken to appreciate that $\left(\sqrt{-x}\right)^2 = |x| = -x$ since $x < 0$.

$$
\begin{aligned}
u_y &= u_\xi \xi_y + u_\eta \eta_y = \left(\partial_\xi + \partial_\eta\right) u, \\
u_{yy} &= \left(\partial_\xi + \partial_\eta\right)^2 u = \left(\partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}\right) u.
\end{aligned}
$$

Thus (1) maps to

$$
-x\left(\partial_{\xi\xi} - 2\partial_{\xi\eta} + \partial_{\eta\eta}\right) u - \frac{1}{2\sqrt{-x}}\left(\partial_\xi - \partial_\eta\right) u + x\left(\partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}\right) u = 0,
$$

which reduces to the $H1$ canonical form

$$
u_{\xi\eta} + \frac{1}{6\left(\eta - \xi\right)}\left(\partial_\xi - \partial_\eta\right) u = 0.
$$

It is not required that one derive the $H2$ canonical form.

Note: because of the confusion I created whether it was question 3.3.2 or 3.3.3 in this assignment, if the student's assignment contains either 3.3.2 or 3.3.3, the question will be graded. If the student's assignment contains both questions, both will be graded and the higher of the two will be used to compute the assignment final grade. Consequently, the solution to both 3.3.2 and 3.3.3 is included here.

3.3.2: (a) The pde

$$
u_{x_1 x_3} = \frac{\partial^2 u}{\partial x_1 \partial x_3} = 0, \tag{1}
$$

may be written in the matrix form

$$
\left(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}\right) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} u = 0.
$$

For convenience let A be the matrix

$$
A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
$$

The eigenvalues of A are determined by

$$
|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda\left(1 - \lambda^2\right) = 0 \Longrightarrow \lambda = \{-1, 0, 1\}.
$$

Since $\lambda = 0$ is an eigenvalue, the pde is parabolic.
(b) To put the pde into canonical form we must first determine the normalized right eigenvectors for each eigenvalue. Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ be the right eigenvectors associated with $\lambda = -1$, 0 and 1, respectively. Thus,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{r}_1 = \mathbf{0} \Longrightarrow \mathbf{r}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{r}_2 = \mathbf{0} \Longrightarrow \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{r}_1 = \mathbf{0} \Longrightarrow \mathbf{r}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, we introduce the new independent variables

$$\xi_1 = \tfrac{1}{\sqrt{2}} (1, 0, -1) \cdot (x_1,\, x_2,\, x_3) = \frac{x_1 - x_3}{\sqrt{2}},$$

$$\xi_2 = (0, 1, 0) \cdot (x_1,\, x_2,\, x_3) = x_2,$$

$$\xi_3 = \tfrac{1}{\sqrt{2}} (1, 0, 1) \cdot (x_1,\, x_2,\, x_3) = \frac{x_1 + x_3}{\sqrt{2}}.$$

The derivatives map according to

$$u_{x_1} = u_{\xi_1} \frac{\partial \xi_1}{\partial x_1} + u_{\xi_2} \frac{\partial \xi_2}{\partial x_1} + u_{\xi_1} \frac{\partial \xi_2}{\partial x_1} = \tfrac{1}{\sqrt{2}} \left(\partial_{\xi_1} + \partial_{\xi_3} \right) u,$$

$$u_{x_2} = u_{\xi_1} \frac{\partial \xi_1}{\partial x_2} + u_{\xi_2} \frac{\partial \xi_2}{\partial x_2} + u_{\xi_1} \frac{\partial \xi_2}{\partial x_2} = u_{\xi_2},$$

$$u_{x_3} = u_{\xi_1} \frac{\partial \xi_1}{\partial x_3} + u_{\xi_2} \frac{\partial \xi_2}{\partial x_3} + u_{\xi_1} \frac{\partial \xi_2}{\partial x_3} = \tfrac{1}{\sqrt{2}} \left(-\partial_{\xi_1} + \partial_{\xi_3} \right) u,$$

$$\Longrightarrow \frac{\partial^2 u}{\partial x_1 \partial x_3} = \tfrac{1}{2} \left(\frac{\partial^2}{\partial \xi_3^2} - \frac{\partial^2}{\partial \xi_1^2} \right) u,$$

so that the pde (1) in canonical form is given by

$$\frac{\partial^2 u}{\partial \xi_1^2} - \frac{\partial^2 u}{\partial \xi_3^2} = 0. \qquad (2)$$

One might think that the pde in (2) is hyperbolic. But it isn't. The reason is that the coefficient of the $\partial^2 u / \partial \xi_2^2$ term is 0, which makes it parabolic.

3.3.3: (a) The pde is given by

$$u_{xx} + 2u_{yz} + \cos(x)\, u_z - \exp\left(y^2\right) u = \cosh(z),$$

which can be written in the matrix form

$$\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} A \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u$$

$$+ \begin{bmatrix} 0 & 0 & \cos x \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u - \exp\left(y^2\right) u = \cosh(z),$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^3$ of A are given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -(\lambda - 1)^2 (\lambda + 1) = 0$$

$$\Longrightarrow \lambda_{1,2} = 1 > 0, \text{ and } \lambda_3 = -1 < 0.$$

Since two of the eigenvalues are positive and the third negative, we conclude that the pde is *hyperbolic*.
(b) The pde is given by

$$u_{xx} + 2u_{xy} + u_{yy} + 2u_{zz} - (1 + xy) u = 0,$$

which can be written in the matrix form

$$\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} A \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u = (1 + xy) u,$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^3$ of A are given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = \lambda (2 - \lambda)^2 = 0$$

$$\Longrightarrow \lambda_{1,2} = 2, \text{ and } \lambda_3 = 0.$$

Since one of the eigenvalues is zero, we conclude that the pde is *parabolic*.
(c) The pde is given by

$$7u_{xx} - 10u_{xy} - 22u_{yz} + u_{yy} - 16u_{xz} - 5u_{zz} = 0,$$

which can be written in the matrix form

$$\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} A \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u = 0,$$

where

$$A = \begin{bmatrix} 7 & -5 & -8 \\ -5 & 1 & -11 \\ -8 & -11 & -5 \end{bmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^3$ of A are given by

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & -5 & -8 \\ -5 & 1-\lambda & -11 \\ -8 & -11 & -5-\lambda \end{vmatrix} = (9-\lambda)\left(\lambda^2 + 6\lambda - 189\right) = 0,$$

which implies

$$\begin{aligned} \lambda_1 &= 9 > 0, \\ \lambda_2 &= -3 + 3\sqrt{22} > 0, \\ \lambda_3 &= -3 - 3\sqrt{22} < 0, \end{aligned}$$

Since two of the eigenvalues are positive and the third is negative, we conclude that the pde is *hyperbolic*.
(d) The pde is given by

$$\exp(z)\, u_{xy} - u_{xx} = \log\left(x^2 + y^2 + z^2 + 1\right),$$

which can be written in the matrix form

$$\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} A \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u = \log\left(x^2 + y^2 + z^2 + 1\right),$$

where

$$A = \begin{bmatrix} -1 & \exp(z)/2 & 0 \\ \exp(z)/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^3$ of A are given by

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -1-\lambda & \exp(z)\,2 & 0 \\ \exp(z)/2 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \\ &= \lambda\left[\exp(2z)/4 - \lambda(1+\lambda)\right] = 0 \\ &\Longrightarrow \lambda_1 = 0,\ \lambda_{2,3} = \frac{-1 \pm \sqrt{1 + \exp(2z)}}{2}. \end{aligned}$$

Since one of the eigenvalues is zero, we conclude that the pde is *parabolic*.

3.3.6: The pde is given by

$$u_{xx} - 2x^2 u_{xz} + u_{yy} + u_{zz} = 0,$$

which can be written in the matrix form

$$\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} A \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u = 0,$$

where

$$A = \begin{bmatrix} 1 & 0 & -x^2 \\ 0 & 1 & 0 \\ -x^2 & 0 & 1 \end{bmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^3$ of A are given by

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 & -x^2 \\ 0 & 1-\lambda & 0 \\ -x^2 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(1-\lambda)^2 - x^4\right] = 0$$

$$\Longrightarrow \lambda_1 = 1 > 0,\ \lambda_2 = 1 + x^2 > 0 \text{ and } \lambda_3 = 1 - x^2.$$

Hence, we conclude that

If $|x| < 1$, the pde is *elliptic*,

If $|x| = 1$, the pde is *parabolic*,

If $|x| > 1$, the pde is *hyperbolic*.