MATH 436: Solutions for Problem Set 2

2.2.19: The pde and initial condition are given by

$$tv_x + xv_t = cv \text{ with } v(x,x) = f(x).$$

To begin, we determine the characteristics. These are the level curves associated with the solutions to

$$\frac{dt}{dx} = \frac{x}{t},$$

which are given by

$$x^2 - t^2 = \beta \ (\beta \text{ is an arbitrary constant}).$$

That is, the characteristics are hyperbolae. The initial data curve $x = t$ is a characteristic associated with $\beta = 0$. Thus, indeed, we are considering a *potential* characteristic initial value problem.

Hence, if a solution exists, then it necessarily follows that

$$\frac{df}{dx} = \frac{dv(x,x)}{dx} = v_x + v_t = \frac{xv_x + xv_t}{x} = \frac{cf}{x} \Longrightarrow f(x) = f_0\, x^c, \tag{1}$$

where f_0 is an arbitrary constant.

To show that if $f(x)$ satisfies (1) then there are infinitely many solutions we find the general solution. We introduce the change of variables

$$\xi = x^2 - t^2 \text{ and } \eta = x \Longleftrightarrow t = \sqrt{\eta^2 - \xi} \text{ and } x = \eta,$$

(we will assume $t \geq 0$) into the pde. NOTE: η can be *any* variable that is *independent* of ξ (we have chosen $\eta = x$ out of convenience). It follows that

$$v_x = v_\xi \xi_x + v_\eta \eta_x = 2xv_\xi + v_\eta \text{ and } v_t = v_\xi \xi_t + v_\eta \eta_t = -2tv_\xi$$

$$\Longrightarrow tv_x + xv_t = t(2xv_\xi + v_\eta) + x(-2tv_\xi) = tv_\eta = v_\eta\sqrt{\eta^2 - \xi},$$

which when substituted into the pde yields

$$v_\eta\sqrt{\eta^2-\xi} = cv \Longrightarrow \frac{dv}{v} = c\frac{d\eta}{\sqrt{\eta^2-\xi}} \Longrightarrow \ln\left(\frac{v}{\phi(\xi)}\right) = c\ln\left(\eta + \sqrt{\eta^2-\xi}\right)$$

$$\Longrightarrow v = \phi(\xi)\left(\eta + \sqrt{\eta^2-\xi}\right)^c,$$

where $\phi(\xi)$ is an arbitrary function of the variable ξ. It therefore follows that the general solution to the pde can be written in the form

$$v(x,t) = \phi\left(x^2 - t^2\right)(x+t)^c. \tag{2}$$

Observe that it follows from (2) that on the initial data curve $x = t$

$$v(x,x) = \phi(0)(2x)^c,$$

which is precisely of the form (1). Since $\phi(\xi)$ is an arbitrary function of the variable ξ, there are infinitely many solutions of the characteristic initial value problem provided (1) holds (i.e., where $f_0 = 2^c\phi(0)$).

2.2.26: The pde and initial condition are given by

$$v_x + v_y + v_z = 0 \text{ with } v(x, y, 0) = f(x, y).$$

This is a initial-value problem in $\mathbb{R}^3$ for which the initial data domain is $\mathbb{R}^2$. Thus, the initial condition corresponds to a surface. In this case the initial data surface is the plane $z = 0$. The Method of Characteristics proceeds as follows. We parameterize the initial data "surface" by introducing the representation

$$x = \lambda,\ y = \tau \text{ and } z = 0,$$

with $(x, y) \in \mathbb{R}^2$. The characteristic equations will be given by

$$\frac{dx}{ds} = 1 \text{ subject to } x|_{s=0} = \lambda,$$

$$\frac{dy}{ds} = 1 \text{ subject to } y|_{s=0} = \tau,$$

$$\frac{dz}{ds} = 1 \text{ subject to } z|_{s=0} = 0,$$

$$\frac{dv}{ds} = 0 \text{ subject to } v|_{s=0} = f(\lambda, \tau).$$

The solution of the characteristic equations is given by

$$x = s + \lambda,\ y = s + \tau \text{ and } z = s \Longrightarrow \lambda = x - z \text{ and } \tau = y - z,$$

$$v = f(\lambda, \tau) \Longrightarrow v(x, y, z) = f(x - z, y - z).$$

2.3.3: The pde and initial condition are

$$u_t + uu_x = -cu,\ \ -\infty < x < \infty,\ t > 0,$$

$$u(x, 0) = ax,\ \ -\infty < x < \infty,$$

where $c > 0$. The initial data curve can be written parametrically in the form $x = \tau$ and $t = 0$ with $\tau \in \mathbb{R}$. The characteristic equations are

$$\frac{dt}{ds} = 1 \text{ subject to } t|_{s=0} = 0 \Longrightarrow t = s, \tag{1}$$

$$\frac{du}{ds} = -cu \text{ subject to } u|_{s=0} = a\tau \Longrightarrow u = a\tau\exp(-cs), \tag{2}$$

$$\frac{dx}{ds} = u = a\tau\exp(-cs) \text{ subject to } x|_{s=0} = \tau$$

$$\Longrightarrow x=\tau\left\{1+\frac{a\left[1-\exp\left(-cs\right)\right]}{c}\right\}. \tag{3}$$

It follows from (1) and (3) that the "inverse relations" are

$$s=t \text{ and } \tau=\frac{cx}{c+a\left[1-\exp\left(-ct\right)\right]},$$

so that $u\left(x,t\right)$ is given by

$$u\left(x,t\right)=\frac{c\,x\exp\left(-ct\right)}{c+a\left[1-\exp\left(-ct\right)\right]}=\frac{acx}{\left(c+a\right)\exp\left(ct\right)-a}.$$

A shock will form the first time

$$\left|u_x\right|=\left|\frac{ac}{\left(c+a\right)\exp\left(ct\right)-a}\right|\to\infty,$$

i.e., when

$$\exp\left(-ct\right)=\frac{a+c}{a}=1+\frac{c}{a}.$$

But $0\leq\exp\left(-ct\right)\leq 1$ since $c>0$ and $t\geq 0$, so that a shock will form only if $-1\leq c/a\leq 0$. And since $c>0$ this means that a shock will form only if $a<-c$ and a shock will never form if $a\geq -c$. If $a<-c$, the shock will form at the time $t=t_s$ given by

$$\exp\left(-ct_s\right)=\frac{a+c}{a}\Longrightarrow t_s=\frac{1}{c}\ln\left(\frac{a}{a+c}\right)>0.$$

2.3.6: The pde and initial condition are given by

$$u_t+cu_x=-u^2,\ \ -\infty<x<\infty,\ t>0,\ c\in\mathbb{R},$$

$$u\left(x,0\right)=x,\ \ -\infty<x<\infty.$$

The initial data curve can be written in the parametric form $x=\tau$ and $t=0$ with $\tau\in\mathbb{R}$. The characteristic equations are given by

$$\frac{dt}{ds}=1 \text{ subject to } \left.t\right|_{s=0}=0\Longrightarrow t=s,$$

$$\frac{dx}{ds}=c \text{ subject to } \left.x\right|_{s=0}=\tau\Longrightarrow x=\tau+cs\ \left(\Longleftrightarrow \tau=x-ct\right),$$

$$\frac{du}{ds}=-u^2 \text{ subject to } \left.u\right|_{s=0}=\tau. \tag{1}$$

The solution to (1) is given by

$$-\frac{du}{u^2}=ds\Longrightarrow\frac{1}{u}-\frac{1}{\tau}=s\Longrightarrow u=\frac{\tau}{1+s\tau}$$

$$\Longrightarrow u\left(x,t\right)=\frac{x-ct}{1+t\left(x-ct\right)}.$$