## MATH 436: Solutions for Problem Set 1

*2.1.1*: The wave equation (2.1.1) is

$$v_{tt} - \gamma^2 v_{xx} = 0.$$

If we define

$$u = v_t \text{ and } w = v_x,$$

then, assuming that $v_{xt} = v_{tx}$ (equal mixed partials), it follows that

$$w_t = u_x,$$

and the pde (2.1.1) itself can be re-written in the form

$$u_t = \gamma^2 w_x.$$

*2.1.4*: The Cauchy-Riemann equations (2.1.6) are

$$w_x = u_y \text{ and } w_y = -u_x.$$

It follows (assuming equal mixed partials) that

$$w_{xx} = u_{yx} = u_{xy} = -w_{yy} \Longrightarrow w_{xx} + w_{yy} = 0,$$

$$u_{xx} = -w_{yx} = -w_{xy} = -u_{yy} \Longrightarrow u_{xx} + u_{yy} = 0.$$

*2.2.1* The pde and initial condition are given by

$$v_t + cv_x + \lambda v = 0, \;\; -\infty < x < \infty, \; t > 0, \; \lambda > 0,$$

$$v(x, 0) = F(x), \;\; -\infty < x < \infty.$$

The initial data curve can be written in the parametric form $x = \tau$ and $t = 0$ with $\tau \in \mathbb{R}$. The characteristic equations are given by

$$\frac{dt}{ds} = 1 \text{ subject to } \left. t \right|_{s=0} = 0, \tag{1}$$

$$\frac{dx}{ds} = c \text{ subject to } \left. x \right|_{s=0} = \tau, \tag{2}$$

$$\frac{dv}{ds} = -\lambda v \text{ subject to } \left. v \right|_{s=0} = F(\tau). \tag{3}$$

Integrating (1) and (2) with respect to $s$ yields, respectively,

$$t = s + \phi(\tau) \text{ and } \left. t \right|_{s=0} = 0 \Longrightarrow \phi = 0 \Longrightarrow t = s,$$

$$x = cs + \eta(\tau) \text{ and } \left. x \right|_{s=0} = \tau \Longrightarrow \eta = \tau \Longrightarrow x = \tau + cs,$$

with the "inverse relations"

$$s = t \text{ and } \tau = x - ct. \tag{4}$$

Therefore, the characteristics are the straight lines in $(x, t)$-space $x - ct = \tau$ (with a specific characteristic determined by a specific value of $\tau$). Consequently, it follows from (3) and (4) that

$$v = F(\tau) e^{-\lambda s} \Longrightarrow v(x,t) = F(x - ct) e^{-\lambda t}.$$

*2.2.2* (a) The pde and initial condition are

$$v_t + cv_x = f(x,t), \quad -\infty < x < \infty, \ t > 0,$$

$$v(x,0) = F(x), \quad -\infty < x < \infty.$$

The initial data curve can be written parametrically in the form $x = \tau$ and $t = 0$ with $\tau \in \mathbb{R}$. The characteristic equations are

$$\frac{dt}{ds} = 1 \text{ subject to } \left. t \right|_{s=0} = 0, \tag{1}$$

$$\frac{dx}{ds} = c \text{ subject to } \left. x \right|_{s=0} = \tau, \tag{2}$$

$$\frac{dv}{ds} = f[x(s,\tau), t(s,\tau)] \text{ subject to } \left. v \right|_{s=0} = F(\tau). \tag{3}$$

Integrating (1) and (2) with respect to $s$ yields, respectively,

$$t = s + \phi(\tau) \text{ and } \left. t \right|_{s=0} = 0 \Longrightarrow \phi = 0 \Longrightarrow t = s,$$

$$x = cs + \eta(\tau) \text{ and } \left. x \right|_{s=0} = \tau \Longrightarrow \eta = \tau \Longrightarrow x = \tau + cs,$$

with the "inverse relations"

$$s = t \text{ and } \tau = x - ct. \tag{4}$$

Therefore, the characteristics are the straight lines in $(x, t)$-space $x - ct = \tau$ (with the particular characteristic determined by the value of $\tau$). It follows from (3) and (4) that

$$v = F(\tau) + \int_0^s f(\tau + c\xi, \xi) \, d\xi = F(x - ct) + \int_0^t f[x + c(\xi - t), \xi] \, d\xi. \tag{5}$$

(b) If $f(x,t) = xt$ and $F(x) = \sin(x)$, it follows from (5) that

$$v(x,t) = \sin(x - ct) + \int_0^t (x - ct)\xi + c\xi^2 \, d\xi$$

$$= \sin(x - ct) + \frac{(x - ct)t^2}{2} + \frac{ct^3}{3} = \sin(x - ct) + \frac{xt^2}{2} - \frac{ct^3}{6}.$$

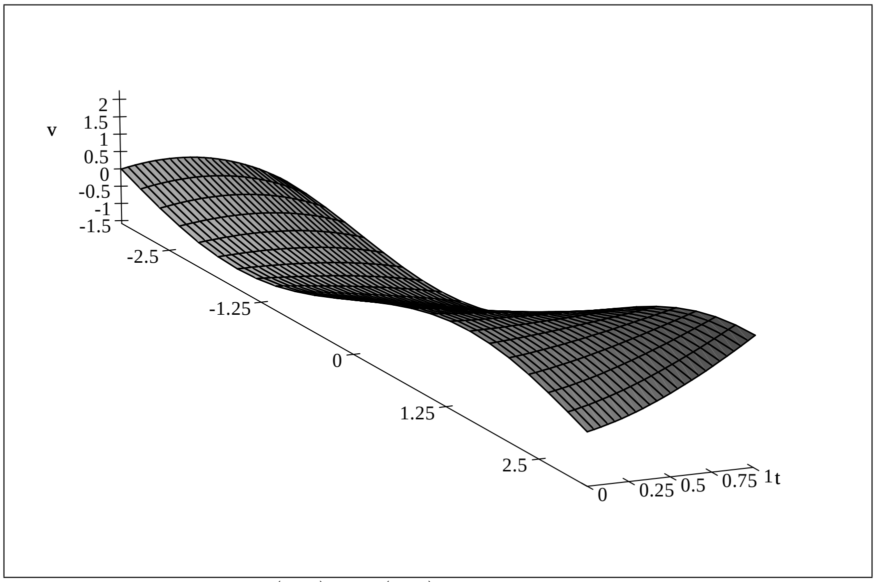


Fig. 1. Plot of $v(x,t)$ *vs.* $(x,t)$ for $-\pi \le x \le \pi$ and $0 \le t \le 1$.

*2.2.5*: The pde and *boundary* condition are given by, respectively,

$$v_t + cv_x = 0, \ \ -\infty < t < \infty, \ x > 0,$$

$$v(0,t) = G(t), \ \ -\infty < x < \infty.$$

The boundary data curve (which is the $t$-axis) can be written in the parametric form $x = 0$ and $t = \tau$ with $\tau \in \mathbb{R}$. The characteristic equations are therefore given by

$$\frac{dt}{ds} = 1 \text{ subject to } \left. t\right|_{s=0} = \tau, \tag{1}$$

$$\frac{dx}{ds} = c \text{ subject to } \left. x\right|_{s=0} = 0, \tag{2}$$

$$\frac{dv}{ds} = 0 \text{ subject to } \left. v\right|_{s=0} = G(\tau). \tag{3}$$

Integrating (1) and (2) with respect to $s$ yields, respectively,

$$t = s + \phi(\tau) \ \text{ and } \ \left. t\right|_{s=0} = \tau \Longrightarrow \phi = \tau \Longrightarrow t = s + \tau,$$

$$x = cs + \eta(\tau) \ \text{ and } \ \left. x\right|_{s=0} = 0 \Longrightarrow \eta = 0 \Longrightarrow x = cs,$$

with the "inverse relations"

$$s = x/c \text{ and } \tau = t - x/c. \tag{4}$$

Therefore, the characteristics are the straight lines in $(x,t)$-space $t - x/c = \tau$ (with a specific characteristic determined by a specific value of $\tau$). Consequently, it follows from (3) and (4) that

$$v = G(\tau) \Longrightarrow v(x,t) = G(t - x/c). \tag{5}$$