Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Consider the 1^{st}-order pde

$$\sin(x)\,u_x + y\cos(x)\,u_y = -\cos(x)\,u^2. \tag{1}$$

   (a) Show that the characteristic curves associated with (1) are given by

$$\xi = \frac{\sin x}{y},$$

   for constant ξ.

   (b) By introducing an appropriate change of independent variables, show that the general solution to (1) can be written in the form

$$u(x,y) = \frac{1}{\ln(\sin x) + \phi\left(\frac{\sin x}{y}\right)}, \tag{2}$$

   where ϕ is an arbitrary function of its argument.

   (c) Assuming the initial condition

$$u(x,1) = \csc x,$$

   use (2) to show that

$$u(x,y) = \frac{y}{y\ln y + \sin x}.$$

2. Show that the solution to the Cauchy problem

$$u_{tt} - u_{xx} = h(x,t), \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) \ \text{ and } \ u_t(x,0) = g(x), \quad -\infty < x < \infty,$$

   where $h(x,t)$, $f(x)$ and $g(x)$ are smooth and spatially square-integrable functions is unique. HINT: You may assume that $u(x,t)$ and all its derivatives are square-integrable functions with respect to x.

3. Suppose the initial data for the linear $1st$-order partial differential equation

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y),$$

   is given by

$$u(x,h(x)) = f(x),$$

   where $y = h(x)$ is a characteristic. Show that, if a solution exists, then

$$\frac{df}{dx} = \frac{c(x,h)\,f(x) + d(x,h)}{a(x,h)}.$$

4. Consider Euler's equations for the isentropic flow of a gas, written in the form

$$u_t + u\,u_x = 0,$$

$$\rho_t + u\,\rho_x + \rho\,u_x = 0,$$

for $-\infty < x < \infty$, $t > 0$ (where u is the velocity and ρ is the density) with the initial conditions

$$u\,(x, 0) = f\,(x) \text{ and } \rho\,(x, 0) = g\,(x)\,, \text{ for } -\infty < x < \infty.$$

(a) Use the Method of Characteristics to show that the solution can be written in the form

$$u = f\,(x - ut) \text{ and } \rho = \frac{g\,(x - ut)}{1 + t\,f'\,(x - ut)}.$$

HINT: Setup the Characteristic Equations and first solve for u and then for ρ.

(b) Assuming that a shock forms the first time, denoted as $t_s > 0$, that $|u_x| \to \infty$, show that

$$t_s = \min_{\tau} \frac{-1}{f'\,(\tau)}, \text{ where } f'\,(\tau) < 0, \tag{3}$$

and that the location of first shock formation, denoted as x_s, is given by

$$x_s = \tau_{\min} + t_s\,f\,(\tau_{\min})\,,$$

where $\tau_{\min}$ is the minimizer associated with (3). HINT: Re-write the solution from Part (a) *implicitly* using the characteristic variable τ.

(c) Show that, assuming $g\,(\tau)$ is bounded for all τ, that

$$\lim_{t \to (t_s)^-} |\rho| = \infty,$$

along the characteristic $\tau = \tau_{\min}$.