Solutions for Math 436 2021 Midterm

Question 1: The pde is given by

$$\sin(x)\, u_x + y\cos(x)\, u_y = -\cos(x)\, u^2.$$

Part (a). The characteristics are the level curves associated with the solution to the ode

$$\frac{dy}{dx} = \frac{y\cos x}{\sin x} \Longrightarrow \int \frac{dy}{y} = \int \frac{\cos x}{\sin x} dx$$

$$\Longrightarrow \ln y + \ln \xi = \ln(\sin x) \Longrightarrow \xi = \frac{\sin x}{y},$$

for constant ξ.
Part (b). To find the general solution we transform from (x, y) to (ξ, η) variables where ξ is the characteristic variable and η is any other independent variable, say,

$$\xi = \frac{\sin x}{y} \text{ and } \eta = x.$$

It follows that

$$u_x = u_\xi \xi_x + u_\eta \eta_x = \frac{\cos x}{y} u_\xi + u_\eta,$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = -\frac{\sin x}{y^2} u_\xi.$$

Substitution into the pde yields

$$\frac{\sin(x)\cos(x)}{y} u_\xi + \sin(x)\, u_\eta - \frac{\sin(x)\cos(x)}{y} u_\xi = \sin(x)\, u_\eta = -\cos(x)\, u^2$$

$$\Longleftrightarrow u_\eta = -\frac{\cos\eta}{\sin\eta} u^2 \Longrightarrow -\int \frac{du}{u^2} = \int \frac{\cos\eta}{\sin\eta} d\eta \Longrightarrow \frac{1}{u} = \ln(\sin\eta) + \phi(\xi)$$

$$\Longleftrightarrow u = \frac{1}{\ln(\sin\eta) + \phi(\xi)} \Longrightarrow u(x, y) = \frac{1}{\ln(\sin x) + \phi\left(\frac{\sin x}{y}\right)}.$$

where $\phi(\xi)$ is an arbitrary function of its argument.
Part (c). If $u(x, 1) = \csc x$, it follows from the general solution that

$$\csc x = \frac{1}{\ln(\sin x) + \phi(\sin x)} \Longrightarrow \sin x = \ln(\sin x) + \phi(\sin x)$$

$$\Longrightarrow \phi(*) = * - \ln(*),$$

and thus

$$u(x, y) = \frac{1}{\ln(\sin x) + \frac{\sin x}{y} - \ln\left(\frac{\sin x}{y}\right)} = \frac{1}{\frac{\sin x}{y} + \ln y} = \frac{y}{y\ln y + \sin x}.$$

Question 2: The pde and initial conditions are given by

$$u_{tt} - u_{xx} = h(x,t), \quad -\infty < x < \infty, \; t > 0,$$

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x), \quad -\infty < x < \infty,$$

where $h(x,t)$, $f(x)$ and $g(x)$ are smooth and spatially square-integrable functions. To show uniqueness, we assume that there are two solutions, given by $u_1(x,t)$ and $u_2(x,t)$, i.e.,

$$(\partial_{tt} - \partial_{xx})\, u_1 = h, \quad -\infty < x < \infty, \; t > 0,$$

$$u_1(x,0) = f \text{ and } \partial_t u_1(x,0) = g, \quad -\infty < x < \infty,$$

and

$$(\partial_{tt} - \partial_{xx})\, u_2 = h, \quad -\infty < x < \infty, \; t > 0,$$

$$u_2(x,0) = f \text{ and } \partial_t u_2(x,0) = g, \quad -\infty < x < \infty.$$

Let $\Phi(x,t) = u_1(x,t) - u_2(x,t)$. We will show that $\Phi(x,t) = 0$ for all $t \geq 0$. Hence $u_1(x,t) = u_2(x,t)$ and we have established uniqueness. It follows that

$$(\partial_{tt} - \partial_{xx})\, \Phi = 0, \quad -\infty < x < \infty, \; t > 0, \tag{1}$$

$$\Phi(x,0) = 0 \text{ and } \Phi_t(x,0) = 0, \quad -\infty < x < \infty. \tag{2}$$

The energy equation associated is obtained by multiplying (1) by Φ_t and rewriting the resulting equation as a space-time divergence, i.e.,

$$\Phi_t(\partial_{tt} - \partial_{xx})\, \Phi = \frac{1}{2}\partial_t \left[(\Phi_t)^2 + (\Phi_x)^2\right] - \partial_x(\Phi_t \Phi_x) = 0.$$

It therefore follows that

$$\frac{d}{dt}\left(\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \; dx\right) = 2\Phi_t \Phi_x|_{-\infty}^{\infty} = 0, \tag{3}$$

since $\Phi_t \Phi_x \to 0$ as $|x| \to \infty$ since $\Phi_{x,t}$ are smooth square-integrable functions. Thus, it follows from (3) that

$$\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \; dx = \left[\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \; dx\right]_{t=0} = 0, \tag{4}$$

since $\Phi(x,0) = 0$ ($\Longrightarrow \Phi_x(x,0) = 0$) and $\Phi_t(x,0) = 0$. Further, it then follows from (4) that

$$\Phi_t(x,t) = \Phi_x(x,t) = 0 \text{ for all } t \geq 0 \Longrightarrow \Phi(x,t) = 0 \text{ for all } t \geq 0.$$

So we have proved uniqueness.

Question 3: The pde and initial condition is given by

$$a(x,y)\, u_x + b(x,y)\, u_y = c(x,y)\, u + d(x,y),$$

$$u(x,h(x)) = f(x),$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x,h)}{a(x,h)}.$$

It follows that

$$\frac{df}{dx} = u_x(x,h(x)) + u_y(x,h(x))\frac{dh}{dx}$$

$$= u_x(x,h(x)) + \frac{b(x,h)}{a(x,h)}u_y(x,h(x)) = \frac{a(x,h)\,u_x(x,h(x)) + b(x,h)\,u_y(x,h(x))}{a(x,h)}$$

$$= \frac{c(x,h)\,u(x,h(x)) + d(x,h)}{a(x,h)} = \frac{c(x,h)\,f(x) + d(x,h)}{a(x,h)}.$$

Question 4: The initial-value problem associated with Euler's equations for the isentropic flow of a gas are given by

$$u_t + u\,u_x = 0,$$

$$\rho_t + u\,\rho_x + \rho\,u_x = 0,$$

for $-\infty < x < \infty$, $t > 0$ (where u is the velocity and ρ is the density) with the initial conditions

$$u(x,0) = f(x) \text{ and } \rho(x,0) = g(x), \text{ for } -\infty < x < \infty.$$

Part (a). The initial data curve is along the x-axis, which we may parameterize as $x = \tau \in \mathbb{R}$ and $t = 0$. The characteristic equations are therefore

$$\frac{dt}{ds} = 1 \text{ subject to } t|_{s=0} = 0,$$

$$\frac{dx}{ds} = u \text{ subject to } x|_{s=0} = \tau,$$

$$\frac{du}{ds} = 0 \text{ subject to } u|_{s=0} = f(\tau),$$

$$\frac{d\rho}{ds} = -\rho\,u_x \text{ subject to } \rho|_{s=0} = g(\tau).$$

The equation for u can be integrated to yield

$$u = f(\tau), \tag{5}$$

which allows the equation for x to be integrated to yield

$$x = sf(\tau) + \tau, \tag{6}$$

and the solution for t is given by

$$t = s.$$

Thus we may write the solution for $u(x,t)$ in the implicit form

$$u = f(\tau) \text{ with } \tau = x - tf(\tau), \tag{7}$$

with τ the characteristic variable or, equivalently, in the form

$$u = f(x - u\,t).$$

In order to solve for ρ we need to compute u_x (as a function of s and τ) and substitute into the characteristic equation for ρ. Thus, from (5) and (6) we have

$$u_x = f'(\tau)\,\tau_x = \frac{f'(\tau)}{1 + s\,f'(\tau)},$$

where prime means differentiation with respect to the argument. Substitution into the characteristic equation for ρ leads to

$$\frac{d\rho}{ds} = -\frac{\rho\,f'(\tau)}{1 + sf'(\tau)} \Longrightarrow \int \frac{d\rho}{\rho} = -\int \frac{f'(\tau)}{1 + sf'(\tau)}\,ds$$

$$\Longrightarrow \ln \rho - \ln [g(\tau)] = -\ln [1 + s\,f'(\tau)] \Longrightarrow \rho = \frac{g(\tau)}{1 + s\,f'(\tau)}.$$

Finally, substituting $s = t$ and $\tau = x - ut$ into this expression allows us to write the solution in the required form

$$u = f(x - ut),$$

$$\rho = \frac{g(x - ut)}{1 + t\,f'(x - ut)}.$$

Part (b). To determine when and where $|u_x| \to \infty$, we start from (7) from which it follows that

$$u_x = f'(\tau)\,\tau_x = \frac{f'(\tau)}{1 + tf'(\tau)},$$

so that, assuming that $f'(\tau)$ is bounded for all τ, $|u_x| \to \infty$ requires

$$1 + tf'(\tau) = 0 \Longrightarrow t = \frac{-1}{f'(\tau)}.$$

The first *positive time* that this happens, denoted by t_s, is therefore given by

$$t_s = \min_{\tau} \frac{-1}{f'(\tau)} \text{ subject to } f'(\tau) < 0. \tag{8}$$

If we denote the minimizer associated with (8) as $\tau_{\min}$, then

$$t_s = -\frac{1}{f'(\tau_{\min})} \text{ and } x_s = \tau_{\min} + t_s\,f(\tau_{\min}).$$

Part (c). In terms of the characteristic variable τ, the solution for ρ can be written in the form

$$\rho = \frac{g(\tau)}{1 + t\,f'(\tau)} \text{ with } \tau = x - tf(\tau)\,.$$

Consequently along the characteristic $\tau = \tau_{\min}$, we have

$$\rho = \frac{g(\tau_{\min})}{1 + t\,f'(\tau_{\min})},$$

from which it follows that

$$\lim_{t \to (t_s)^-} |\rho| = \lim_{t \to (t_s)^-} \frac{|g(\tau_{\min})|}{|1 + t\,f'(\tau_{\min})|} = |g(\tau_{\min})| \lim_{t \to (t_s)^-} \frac{1}{|1 + t\,f'(\tau_{\min})|} = \infty,$$

assuming $|g(\tau_{\min})| < \infty$.