Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

21 December 2021

Instructions. Please answer all 4 questions. Each question is worth 25 points

1. Classify and reduce to canonical form the scalar pde in $\mathbb{R}^3$ given by

$$u_{xx} - 2u_{xz} + u_{yy} + u_{zz} = 0.$$

2. Consider the 2×2 system of pdes is given by

$$\mathbf{u}_y + A\mathbf{u}_x = \mathbf{0} \text{ where } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

   (a) Classify the system of pdes.

   (b) Reduce the system of pdes to normal or characteristic form.

3. Consider the pde

$$u_{tt} + 2u_{xt} - \beta\, u_{xx} = 0,\ x \text{ and } \beta \in \mathbb{R},\ t > 0.$$

   (a) Determine the stability index Ω as a function of the parameter β.

   (b) Determine the stability of the pde as a function of the parameter β.

   (c) Determine the values of the parameter β for which the Cauchy problem is ill-posed.

4. The linear shallow water equations for a rotating fluid can be written in the form

$$u_t - v = -h_x, \tag{1}$$

$$v_t + u = -h_y, \tag{2}$$

$$h_t + u_x + v_y = 0, \tag{3}$$

   where (1) and (2) are the x and y *momentum equations*, respectively, (3) is the *mass equation*, u and v are the x and y *velocities*, respectively, and h is the deflection of the free surface about its rest position.

   (a) Show that (1) and (2) can be combined together to yield

$$(\partial_{tt} + 1)\, u = -h_y - h_{xt},$$

$$(\partial_{tt} + 1)\, v = h_x - h_{yt}.$$

   (b) Use Part (a) to show that (3) can be written in the form

$$(\partial_{tt} + 1 - \partial_{xx} - \partial_{yy})\, h_t = 0. \tag{4}$$

(c) Consider the "channel" domain

$$G = \{(x, y) \mid -\infty < x < \infty,\, 0 < y < 1\},$$

with the Dirichlet y-boundary conditions

$$h(x, 0, t) = h(x, 1, t) = 0,\ x \in \mathbb{R},\ t > 0.$$

Assuming the *neutrally-stable* along-channel propagating normal mode solution to (4) given by

$$h = a \sin(\pi y) \exp(ikx - i\omega t) + c.c.,$$

where a is the amplitude, k is the real x-direction wavenumber, and ω is the real frequency, show that the *dispersion relation* (for $\omega \neq 0$) is given by

$$\omega = \pm\sqrt{1 + \pi^2 + k^2}.$$

(d) Show that the $\omega \neq 0$ normal modes are *dispersive*.