Solutions for Math 436 2021 Final

Question 1: The pde, given by,

$$u_{xx} - 2u_{xz} + u_{yy} + u_{zz} = 0.$$

can be written in the matrix form

$$\begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix} A \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} u = 0 \text{ where } A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

To classify the pde we compute the eigenvalues $\{\lambda_i\}_{i=1}^3$ of A as follows:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$$

$$\Longrightarrow \lambda_1 = 1 > 0,\ \lambda_2 = 2 > 0 \text{ and } \lambda_3 = 0.$$

Since $\lambda_3 = 0$, the pde is *parabolic*.
To transform the pde into canonical form we first need to compute the orthonormal eigenvectors, denoted by $\mathbf{r}_i$, determined from

$$(A - \lambda_i I)\,\mathbf{r}_i = \mathbf{0},$$

for $i = 1, 2, 3$, respectively. We obtain

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \mathbf{r}_1 = \mathbf{0} \Longrightarrow \mathbf{r}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \mathbf{r}_2 = \mathbf{0} \Longrightarrow \mathbf{r}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \mathbf{r}_3 = \mathbf{0} \Longrightarrow \mathbf{r}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We now introduce the new coordinates or *independent variables* (α, β, μ) given by

$$\alpha = \frac{\mathbf{r}_1^\top \cdot \mathbf{x}}{\sqrt{|\lambda_1|}} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y,$$

$$\beta = \frac{\mathbf{r}_2^\top \cdot \mathbf{x}}{\sqrt{|\lambda_2|}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{x - z}{2},$$

$$\mu = \mathbf{r}_3^\top \cdot \mathbf{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{x+z}{\sqrt{2}}.$$

Consequently, the derivatives transform according to

$$u_x = \frac{1}{2} u_\beta + \frac{1}{\sqrt{2}} u_\mu \Longrightarrow u_{xx} = \frac{1}{4} u_{\beta\beta} + \frac{1}{\sqrt{2}} u_{\beta\mu} + \frac{1}{2} u_{\mu\mu},$$

$$u_z = -\frac{1}{2} u_\beta + \frac{1}{\sqrt{2}} u_\mu \Longrightarrow u_{zz} = \frac{1}{4} u_{\beta\beta} - \frac{1}{\sqrt{2}} u_{\beta\mu} + \frac{1}{2} u_{\mu\mu} \text{ and } u_{xz} = -\frac{1}{4} u_{\beta\beta} + \frac{1}{2} u_{\mu\mu},$$

$$u_{yy} = u_{\alpha\alpha}.$$

Finally, substitution into the pde yields

$$u_{\alpha\alpha} + u_{\beta\beta} = 0.$$

Question 2a: If $y = h(x)$ are the characteristics, then $h(x)$ must satisfy

$$|I - h'(x) A| = \begin{vmatrix} 1-h' & -2h' \\ -2h' & 1-h' \end{vmatrix} = (1-h')^2 - 4(h')^2 = 0$$

$$\Longrightarrow 1 - h' = \pm 2h' \Longrightarrow h' = \frac{1}{3} \text{ or } -1.$$

Since all of the h' are real and distinct, *the system is strictly or totally hyperbolic.*
Question 2b: To reduce to normal or characteristic form we first need to compute the right eigenvectors associated with $1/h'$ as computed in Part a. The *right eigenvectors* of A associated with the eigenvalues $1/h'$, i.e., -1 and 3, and denoted by $\mathbf{r}_{-1}$ and $\mathbf{r}_3$, respectively, are determined by

$$(I + A)\,\mathbf{r}_{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{r}_{-1} = \mathbf{0} \Longrightarrow \mathbf{r}_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\left(I - \tfrac{1}{3}A\right)\mathbf{r}_3 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \mathbf{r}_3 = \mathbf{0} \Longrightarrow \mathbf{r}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that $\mathbf{r}_{-1} \cdot \mathbf{r}_3 = 0$. Thus, introducing the *dependent variable* transformation

$$\mathbf{u} \equiv R\mathbf{v} = R \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \text{ where } R \equiv \begin{bmatrix} \mathbf{r}_{-1} & \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Longrightarrow R^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

into the pde, leads to

$$\mathbf{v}_y + D\mathbf{v}_x = \mathbf{0}, \text{ where } D \equiv R^{-1}AR = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix},$$

or, in component form,

$$\partial_y v_1 - \partial_x v_1 = 0 \text{ and } \partial_y v_2 + 3\partial_x v_2 = 0.$$

Question 3a: The pde is given by

$$u_{tt} + 2u_{xt} - \beta\, u_{xx} = 0.$$

To determine the stability index Ω, we substitute

$$u = a \exp\left(ikx + \lambda t\right) + c.c.,$$

into the the pde to yield

$$\left(\lambda^2 + 2ik\lambda + \beta k^2\right) a \exp\left(ikx + \lambda t\right) + c.c. \ = 0.$$

For non-trivial (i.e., $a \neq 0$) solutions we must have

$$\lambda^2 + 2ik\lambda + \beta k^2 = 0$$

$$\Longrightarrow \lambda = -ik \pm \sqrt{-k^2 - \beta k^2} = -ik \pm i\left|k\right| \sqrt{1+\beta}.$$

It therefore follows that

$$\Omega = \mathrm{lub}_k \,\mathrm{Re}\left[\lambda\left(k\right)\right] = \begin{cases} 0 \text{ if } \beta \geq -1 \\ +\infty \text{ if } \beta < -1. \end{cases}$$

Question 3b: The pde is *neutrally stable* if $\beta \geq -1$ and is *unstable* if $\beta < -1$.
Question 3c: The Cauchy problem is ill-posed if $\beta < -1$.

Question 4a: The linear shallow water equations are given by

$$u_t - v = -h_x, \tag{1}$$

$$v_t + u = -h_y, \tag{2}$$

$$h_t + u_x + v_y = 0, \tag{3}$$

Taking the t-derivative of (1) and (2) yields, respectively,

$$u_{tt} - v_t = -h_{xt},$$

$$v_{tt} + u_t = -h_{yt}.$$

Now if u_t and v_t are eliminated in these two equations using (1) and (2) one obtains, respectively,

$$u_{tt} - \left(-u - h_y\right) = -h_{xt},$$

$$v_{tt} + \left(v - h_x\right) = -h_{yt},$$

which simplifies to

$$\left(\partial_{tt} + 1\right) u = -h_y - h_{xt}, \tag{4}$$

$$\left(\partial_{tt} + 1\right) v = h_x - h_{yt}. \tag{5}$$

Question 4b: It follows from (3) that

$$(\partial_{tt}+1)\,h_t + \left[(\partial_{tt}+1)\,u\right]_x + \left[(\partial_{tt}+1)\,v\right]_y = 0,$$

which if we substitute in (4) and (5) from Part (a) implies that

$$(\partial_{tt}+1)\,h_t - h_{xy} - h_{xxt} + h_{xy} - h_{yyt} = 0,$$

which simplifies to

$$(\partial_{tt}+1-\partial_{xx}-\partial_{yy})\,h_t = 0. \tag{6}$$

Question 4c: Substitution of the *neutrally-stable* along-channel propagating normal mode solution

$$h = a\sin(\pi y)\exp(ikx - i\omega t) + c.c.,$$

into (6) leads to

$$-i\omega\left(-\omega^2+1+k^2+\pi^2\right)a\sin(\pi y)\exp(ikx-i\omega t) + c.c. \ = 0.$$

For a non-trivial solution (i.e., $a \neq 0$) it must follow that

$$\omega\,\left(-\omega^2+1+k^2+\pi^2\right) = 0,$$

which implies that for the $\omega \neq 0$ solutions

$$\omega = \pm\sqrt{1+\pi^2+k^2}.$$

Question 4d: To show that the solution is dispersive we need to show that $dc/dk \neq 0$, where c is the phase velocity. The *phase velocity* c is given by

$$c \equiv \frac{\omega}{k} = \pm\frac{\sqrt{1+\pi^2+k^2}}{k} \Longrightarrow \frac{dc}{dk} = \mp\frac{1+\pi^2}{k^2\sqrt{1+\pi^2+k^2}} \neq 0.$$

Hence the normal modes are dispersive.