Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. (a) Consider the linear 1^{st}-order partial differential equation (pde) in $\mathbb{R}^2$ given by

$$(3y-x)\,u_x+(x+y)\;u_y=-2x\,(3y-x)\,u,\;\;-\infty<x<\infty,\;\;-\infty<y<\infty\;.$$

Show that the characteristics are the hyperbolae given by

$$\xi=x^2+2xy-3y^2,$$

for constant ξ. HINT: The solution to the homogeneous ode $dy/dx=h\,(y/x)$ can be found in the form $y=x\,\phi\,(x)$.

(b) Determine the class of functions $f\,(x)$ for which the pde in Part a, subject to the initial condition

$$u\,(x,x)=f\,(x)\,,$$

is a *characteristic initial value problem.*

(c) By introducing a suitable change of independent variables into the pde in Part a, show that the general solution can be written in the form

$$u\,(x,y)=F\left(x^2+2xy-3y^2\right)\exp\left(-x^2\right),$$

where F is an arbitrary differentiable function of its argument.

2. Suppose that the initial data for the linear 1^{st}-order pde

$$a\,(x,y)\,u_x+b\,(x,y)\,u_y=c\,(x,y)\,u+d\,(x,y)\,,$$

is given by

$$u\,(x,h\,(x))=f\,(x)\,,$$

where $y=h\,(x)$ is a characteristic associated with the pde, and a, b, c and d are all smooth functions. Show that it is impossible to uniquely determine *both* $u_x\,(x,h\,(x))$ and $u_y\,(x,h\,(x))$, if a solution exists at all.

3. (a) Use the *Method of Characteristics* to show that the solution to the damped Burger's equation

$$u_t+u\,u_x=-u/2,\;\;-\infty<x<\infty,\;\;t>0,$$

with initial condition

$$u\,(x,0)=f\,(x)\,,\;\;-\infty<x<\infty,$$

can be written in the implicit form

$$u\,(x,t)=f\,(\tau)\exp\left(-t/2\right),$$

$$x=\tau+2\,f\,(\tau)\;\left[1-\exp\left(-t/2\right)\right].$$

(b) Use the result from Part a to show that the *first time* a shock will form, denoted by $t = t_s > 0$, is given by

$$t_s = \min_{\tau} \left\{ -2 \ln \left[1 + \frac{1}{2f'(\tau)} \right] \right\} > 0,$$

and thus show that a *necessary condition* for a shock to form for finite $t_s > 0$ is that there exist τ satisfying

$$f'(\tau) < -1/2.$$

(c) Assuming that

$$f(x) = \begin{cases} 0 \text{ for } |x| \geq 1, \\ -\sin(\pi x) \text{ for } |x| < 1, \end{cases}$$

show that the time and location (denoted as $x = x_s$) of first shock formation is given by

$$t_s = -2 \ln \left(\frac{2\pi - 1}{2\pi} \right) \text{ and } x_s = 0,$$

respectively. HINT: One will not be able to *explicitly* determine $u(x,t)$ for $|x| < 1$ (but one can for $|x| \geq 1$).

4. Consider the 2^{nd}-order linear pde and initial conditions in $\mathbb{R}^2$ given by

$$u_{xx} + 2u_{xy} + \left(1 - x^2\right) u_{yy} + u_y = 0, \ (x, y) \in \mathbb{R}^2,$$

with

$$u(x, x) = f\left(x^2/2\right), \ -\infty < x < \infty,$$
$$u_x(x, x) + (1 + x) u_y(x, x) = 2x\, g\left(x^2/2\right), \ -\infty < x < \infty,$$

where f and g are assumed smooth integrable functions.

(a) Classify the pde as a function of (x, y). For the region where the pde is *hyperbolic*, show that the *characteristic variables*, denoted as (ξ, η), can be written in the form

$$\xi = y - x + \frac{x^2}{2},$$

$$\eta = y - x - \frac{x^2}{2}.$$

(b) When the pde is hyperbolic, show that the $H1$ *canonical form* of the pde *and* the initial conditions can be written in the form

$$u_{\xi\eta} - \frac{1}{2(\xi - \eta)} u_\xi = 0,$$

with

$$u(\xi, -\xi) = f(\xi),$$
$$u_\xi(\xi, -\xi) = g(\xi).$$

(c) Solve the $H1$ *canonical* pde *and* the initial conditions in Part b, and after transforming back to (x, y) variables, show that the solution $u(x, y)$ to the original pde and initial conditions is given by

$$u(x,y) = f\left(x^2/2 + x - y\right) + \sqrt{2}\int_{x^2/2+x-y}^{y-x+x^2/2} \sqrt{\frac{s}{s + x^2/2 + x - y}}\, g(s)\, ds.$$

HINT: Introduce the *integrating factor* $\sqrt{\xi - \eta}$ into the $H1$ canonical form and first integrate with respect to η.