Solutions for Math 436 2019 Midterm

Question 1 Part a: The characteristics are the levels curves associated with the ode

$$\frac{dy}{dx} = \frac{x+y}{3y-x} = \frac{1+y/x}{3y/x-1}.$$

Following the HINT, the solution can be obtained by assuming

$$y = x\phi\left(x\right) \Longrightarrow \frac{dy}{dx} = \phi + x\frac{d\phi}{dx} \Longrightarrow \phi + x\frac{d\phi}{dx} = \frac{1+\phi}{3\phi-1}$$

$$\Longrightarrow x\frac{d\phi}{dx} = \frac{1+\phi}{3\phi-1} - \phi = \frac{1+2\phi-3\phi^2}{3\phi-1}$$

$$\Longrightarrow \int \frac{3\phi-1}{1+2\phi-3\phi^2}\,d\phi = \int \frac{dx}{x}$$

$$\Longrightarrow -\frac{1}{2}\ln\left(1+2\phi-3\phi^2\right) + \frac{1}{2}\ln\xi = \ln x,$$

$$\Longrightarrow \ln\xi = \ln\left(1+2\phi-3\phi^2\right) + 2\ln x = \ln\left[x^2\left(1+2\phi-3\phi^2\right)\right]$$

$$\Longrightarrow \xi = x^2\left(1+2\phi-3\phi^2\right) = x^2+2xy-3y^2,$$

where ξ is a conveniently written constant of integration.
Alternate Solution: In fact, the way Question 1a was worded means it is not necessary to explicitly solve the ode that describes the characteristics if one argues as follows. We begin as before and state that the characteristics are the levels curves associated with the ode

$$\frac{dy}{dx} = \frac{x+y}{3y-x}.$$

To show that the curves

$$\xi = x^2+2xy-3y^2,$$

for constant ξ corresponds to a level curve of the ode all we need to do is implicitly differentiate the ξ-equation with respect to x holding ξ constant to find

$$0 = 2\left(x+y\right) + \left(2x-6y\right)\frac{dy}{dx},$$

and re-arrange this expression to see that

$$\left(\frac{dy}{dx}\right)_\xi = \frac{x+y}{3y-x}.$$

Thus, we have shown that the curves for which ξ is constant solves the ode that describes the characteristics. QED.
Part b. The initial data curve is given by $y=x$. First, we verify that $y=x$ is a characteristic. We see that $y=x$ is a characteristic associated with $\xi=0$. Second, we must ensure that the initial data $u\left(x,x\right) = f\left(x\right)$ is *consistently* given

on the initial data curve $y = x$. We recall that the initial data $u(x, h(x)) = f(x)$ is consistently given on the characteristic $y = h(x)$ for the pde

$$a(x, y) u_x + b(x, y) u_y = c(x, y) u + d(x, y),$$

if

$$\frac{df}{dx} = \frac{c(x, h) f + d(x, h)}{a(x, h)}.$$

For the pde in question

$$h = x,\ c(x, h) = -2x(3h - x) = -4x^2,\ d = 0,\ a(x, h) = x + h = 2x,$$

which implies

$$\frac{df}{dx} = \frac{-4x^2 f + 0}{2x} = -2xf \Longrightarrow f(x) = C \exp\left(-x^2\right),$$

for any constant C.

Part c. To obtain the general solution to the pde we introduce the change of variable

$$\xi = x^2 + 2xy - 3y^2 \text{ and } \eta = x,$$

where ξ is the characteristic variable and η is any other independent variable (but we choose $\eta = x$). It follows that

$$u_x = u_\xi \xi_x + u_\eta \eta_x = 2(x + y) u_\xi + u_\eta,$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = 2(x - 3y) u_\xi.$$

Substitution into the pde in Part a yields

$$(3y - x)[2(x + y) u_\xi + u_\eta] + 2(x + y)(x - 3y) u_\xi = -2x(3y - x) u$$

$$\Longrightarrow u_\eta = -2\eta u \Longrightarrow u = F(\xi) \exp\left(-\eta^2\right),$$

where $F(\xi)$ is an arbitrary differentiable function of its argument. Thus,

$$u(x, y) = F\left(x^2 + 2xy - 3y^2\right) \exp\left(-x^2\right).$$

Question 2: The pde and initial condition are given by

$$a(x, y) u_x + b(x, y) u_y = c(x, y) u + d(x, y),$$

$$u(x, h(x)) = f(x),$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x, h)}{a(x, h)}.$$

Differentiating the initial condition with respect to x leads to

$$\frac{df}{dx} = u_x\left(x, h\left(x\right)\right) + u_y\left(x, h\left(x\right)\right)\frac{dh}{dx},$$

but we also have, from the pde itself, that on $y = h\left(x\right)$

$$a\left(x,h\right)u_x\left(x,h\left(x\right)\right) + b\left(x,h\right)u_y\left(x,h\left(x\right)\right) = c\left(x,h\right)f\left(x\right) + d\left(x,h\right).$$

These two equations can be written as the 2×2 system

$$\left[\begin{array}{cc} 1 & dh/dx \\ a\left(x,h\right) & b\left(x,h\right)\end{array}\right]\left[\begin{array}{c} u_x\left(x,h\left(x\right)\right) \\ u_y\left(x,h\left(x\right)\right)\end{array}\right] = \left[\begin{array}{c} df/dx \\ c\left(x,h\right)f\left(x\right) + d\left(x,h\right)\end{array}\right].$$

If $u_x\left(x,h\left(x\right)\right)$ and $u_y\left(x,h\left(x\right)\right)$ were *uniquely determined*, assuming a solution exists at all, it would imply that the coefficient matrix

$$\left[\begin{array}{cc} 1 & dh/dx \\ a\left(x,h\right) & b\left(x,h\right)\end{array}\right],$$

was invertible. But

$$\left|\begin{array}{cc} 1 & dh/dx \\ a\left(x,h\right) & b\left(x,h\right)\end{array}\right| = b\left(x,h\right) - a\left(x,h\right)\frac{dh}{dx} = 0,$$

so the inverse of the coefficient matrix doesn't exist implying that $u_x\left(x,h\left(x\right)\right)$ and $u_y\left(x,h\left(x\right)\right)$ *cannot be uniquely determined*.

Question 3: Part a. The initial data curve Γ can parameterized as:

$$\Gamma = \left\{\left(x,t\right) \mid x = \tau,\ t = 0,\ \tau\in\mathbb{R}\right\},$$

with

$$u|_{\Gamma} = f\left(\tau\right).$$

The characteristic equations and their solution are given by

$$\frac{dt}{ds} = 1 \text{ with } t|_{s=0} = 0 \Longrightarrow t = s,$$

$$\frac{du}{ds} = -u/2 \text{ with } u|_{s=0} = f\left(\tau\right) \Longrightarrow u = f\left(\tau\right)\exp\left(-s/2\right),$$

$$\frac{dx}{ds} = u = f\left(\tau\right)\exp\left(-s/2\right) \text{ with } x|_{s=0} = \tau$$

$$\Longrightarrow x = \tau + 2f\left(\tau\right)\left[1 - \exp\left(-s/2\right)\right].$$

It therefore follows that the solution $u\left(x,t\right)$ can be written in the implicit form

$$u\left(x,t\right) = f\left(\tau\right)\exp\left(-t/2\right),$$

$$x = \tau + 2f\left(\tau\right)\left[1 - \exp\left(-t/2\right)\right].$$

Part b. A shock will form when $|u_x| \to \infty$. It follows from the solution in Part a that

$$u_x(x,t) = f'(\tau)\, \tau_x\, \exp(-t/2),$$

and from the characteristic equation that

$$1 = \{1 + 2 f'(\tau) [1 - \exp(-t/2)]\}\, \tau_x \Longrightarrow \tau_x = \frac{1}{1 + 2 f'(\tau) [1 - \exp(-t/2)]}$$

$$\Longrightarrow u_x = \frac{f'(\tau) \exp(-t/2)}{1 + 2 f'(\tau) [1 - \exp(-t/2)]}.$$

Assuming $f'(\tau)$ is bounded, $|u_x| \to \infty$ only if

$$1 + 2 f'(\tau) [1 - \exp(-t/2)] = 0 \Longrightarrow t = -2 \ln \left[1 + \frac{1}{2f'(\tau)}\right]$$

Consequently, the *first time a shock will form*, denoted by $t = t_s > 0$, will be given by

$$t_s = \min_{\tau} \left\{-2 \ln \left[1 + \frac{1}{2f'(\tau)}\right]\right\} > 0.$$

Since we require that $t_s > 0$, it follows that

$$\ln \left[1 + \frac{1}{2f'(\tau)}\right] < 0 \Longrightarrow 0 < 1 + \frac{1}{2f'(\tau)} < 1$$

$$\Longrightarrow -1 < \frac{1}{2f'(\tau)} < 0 \Longrightarrow -f'(\tau) > \frac{1}{2} \Longrightarrow f'(\tau) < -\frac{1}{2}.$$

Part c. Assuming

$$f(\tau) = \begin{cases} 0 \text{ for } |\tau| \geq 1, \\ -\sin(\pi\tau) \text{ for } |\tau| < 1, \end{cases}$$

it follows that

$$f'(\tau) = \begin{cases} 0 \text{ for } |\tau| > 1, \\ -\pi \cos(\pi\tau) \text{ for } |\tau| < 1. \end{cases}$$

We immediately conclude that

$$|\tau| \geq 1 \Longrightarrow f(\tau) = 0 \Longrightarrow x = \tau \Longleftrightarrow \tau = x\ \forall\ t \geq 0$$

$$\Longrightarrow u(x,t) = 0 \text{ for } |x| \geq 1\ \forall\ t \geq 0.$$

This in turn implies that

$$u_x(x,t) = 0 \text{ for } |x| > 1\ \forall\ t \geq 0.$$

Thus, a shock will never develop in the regions $|x| > 1$. Hence a shock can only develop for those characteristics $|\tau| < 1$. Since $f'(\tau) = -\pi \cos(\pi\tau)$ for $|\tau| < 1$, it follows that

$$t_s = \min_{\tau} \left\{-2 \ln \left[1 - \frac{1}{2\pi \cos(\pi\tau)}\right]\right\} > 0,$$

which will be realized by

$$\max_{\tau}\left\{1-\frac{1}{2\pi\cos\left(\pi\tau\right)}\right\},$$

provided

$$0<1-\frac{1}{2\pi\cos\left(\pi\tau_{\max}\right)}<1\Longleftrightarrow 0<\frac{1}{2\pi\cos\left(\pi\tau_{\max}\right)}<1$$

$$\Longleftrightarrow \cos\left(\pi\tau_{\max}\right)>\frac{1}{2\pi}\Longleftrightarrow\left|\tau_{\max}\right|<\arccos\left(1/2\pi\right)/\pi\simeq 0.449.$$

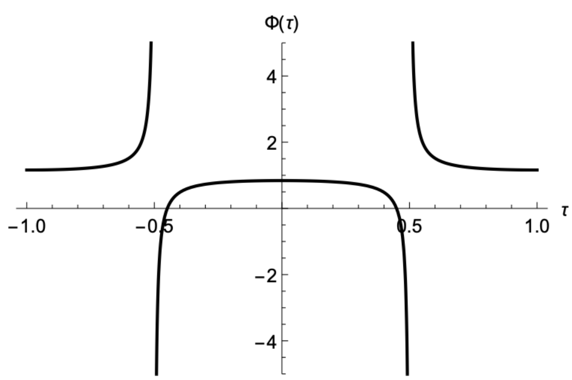
Consequently, the *maximizer*, denoted by $\tau_{\max}$, is immediately seen by inspection to be $\tau_{\max}=0$.
Alternative argument: If we consider the function $\Phi\left(\tau\right)$ defined by

$$\Phi\left(\tau\right)\equiv 1-\frac{1}{2\pi\cos\left(\pi\tau\right)},$$

over the interval $|\tau|<1$, we see that $\Phi\left(\tau\right)$ is continuously differentiable everywhere except at $\tau=\pm 1/2$ where it has vertical asymptotes and that

$$\Phi\left(\tau\right)>1 \text{ for } \frac{1}{2}<|\tau|<1,$$

which would imply that $t_s<0$. Below is a graph of $\Phi\left(\tau\right)$ for $-1<\tau<1$.



Consequently, it must be that $0\leq\left|\tau_{\max}\right|<1/2$ where $\Phi\left(\tau\right)$ *is* continuously differentiable. Hence, $\tau_{\max}$ can be determined from

$$\frac{d}{d\tau}\left(1-\frac{1}{2\pi\cos\left(\pi\tau\right)}\right)=\frac{\sin\left(\pi\tau\right)}{2\cos^2\left(\pi\tau\right)}=0\Longrightarrow\tau_{\max}=0.$$

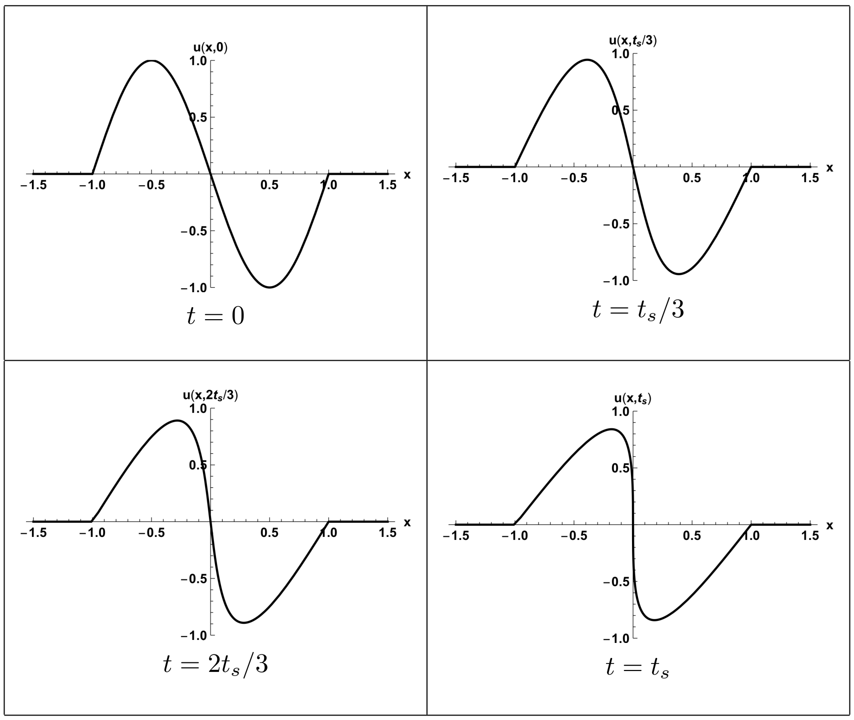
It therefore follows that

$$t_s=-2\ln\left[1-\frac{1}{2\pi\cos\left(\pi\tau_{\max}\right)}\right]=-2\ln\left(\frac{2\pi-1}{2\pi}\right)\simeq 0.3467,$$

and further that

$$x_s = \tau_{\max} + 2\, f\left(\tau_{\max}\right)\left[1 - \exp\left(-t_s/2\right)\right] = 0,$$

since $f\left(\tau_{\max}\right) = -\sin\left(0\right) = 0$. Below, the solution $u\left(x,t\right)$ is shown at $t = 0$, $t_s/3$, $2t_s/3$ and t_s, respectively.



Question 4: The 2^{nd}-order pde and initial conditions are given by

$$u_{xx} + 2u_{xy} + \left(1 - x^2\right) u_{yy} + u_y = 0,\ \left(x,y\right) \in \mathbb{R}^2,$$

with

$$u\left(x,x\right) = f\left(x^2/2\right),\ -\infty < x < \infty,$$

$$u_x\left(x,x\right) + \left(1+x\right) u_y\left(x,x\right) = 2xg\left(x^2/2\right),\ -\infty < x < \infty,$$

where f and g are assumed smooth integrable functions.
Part a. In order to classify the pde we examine the roots of the quadratic

$$\omega^2 + 2\omega + 1 - x^2 = 0$$

$$\Longrightarrow \omega^{+,-} = \frac{-2 \pm \sqrt{4 - 4\left(1 - x^2\right)}}{2} = -1 \pm x.$$

The pde is therefore hyperbolic for $x \neq 0$ and is parabolic for $x = 0$ (along the y-axis).

In the region where the pde is hyperbolic ($x \neq 0$), the characteristics are determined from

$$\left(\frac{dy}{dx}\right)_\xi = -\omega^+ = 1 - x \Longrightarrow y = x - x^2/2 + \xi,$$

$$\left(\frac{dy}{dx}\right)_\eta = -\omega^- = 1 + x \Longrightarrow y = x + x^2/2 + \eta.$$

Thus, the characteristic variables are

$$\xi = y - x + x^2/2,$$

$$\eta = y - x - x^2/2.$$

Part b. To reduce the pde to $H1$ canonical form when it is hyperbolic we must transform the (x, y)-derivatives to (ξ, η)-derivatives. We have

$$u_x = u_\xi \xi_x + u_\eta \eta_x = (x-1)\, u_\xi - (1+x)\, u_\eta,$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta,$$

from which we find that

$$u_{xx} = (1-x)^2\, u_{\xi\xi} + 2\left(1 - x^2\right) u_{\xi\eta} + (1+x)^2\, u_{\eta\eta} + u_\xi - u_\eta,$$

$$u_{xy} = (x-1)\, u_{\xi\xi} - 2u_{\xi\eta} - (1+x)\, u_{\eta\eta},$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Substitution into the pde results in

$$-4x^2 u_{\xi\eta} + 2u_\xi = 0,$$

which can be rearranged into

$$u_{\xi\eta} - \frac{1}{2\,(\xi - \eta)} u_\xi = 0, \tag{1}$$

since $\xi - \eta = x^2 > 0$.

For the initial conditions we observe that

$$u_x + (1+x)\, u_y = (x-1)\, u_\xi - (1+x)\, u_\eta + (1+x)\,(u_\xi + u_\eta) = 2xu_\xi.$$

In addition, we see that the initial data curve $y = x$ results in $\xi = x^2/2 > 0$ and $\eta = -x^2/2 < 0$ which implies $\eta = -\xi$. Hence, the initial data transforms to

$$u\,(\xi, -\xi) = f\,(\xi)\,,$$

$$u_\xi\,(\xi, -\xi) = g\,(\xi)\,,$$

for $\xi > 0$.
Part c. With respect to integrating (1) with respect to η, the integrating factor is determined from

$$\exp\left(-\frac{1}{2}\int\frac{d\eta}{\xi-\eta}\right) = \exp\left(\frac{1}{2}\ln|\xi-\eta|\right) = \sqrt{\xi-\eta},$$

since $\xi - \eta = x^2 > 0$. Thus, from (1) we have

$$\sqrt{\xi-\eta}\,u_{\xi\eta} - \frac{1}{2\sqrt{\xi-\eta}}\,u_\xi = \left(\sqrt{\xi-\eta}\,u_\xi\right)_\eta = 0$$
$$\Longrightarrow \sqrt{\xi-\eta}\,u_\xi(\xi,\eta) = \phi(\xi),$$

where $\phi(\xi)$ is an arbitrary function of its argument. However $u_\xi(\xi,-\xi) = g(\xi)$, which implies

$$\phi(\xi) = \sqrt{2\xi}\,g(\xi).$$

Consequently, we have

$$u_\xi(\xi,\eta) = \sqrt{\frac{2\xi}{\xi-\eta}}\,g(\xi)$$
$$\Longrightarrow u(\xi,\eta) = \psi(\eta) + \sqrt{2}\int_0^\xi \sqrt{\frac{s}{s-\eta}}\,g(s)\,ds,$$

where $\psi(\eta)$ is an arbitrary function of its argument and, without loss of generality, we have taken the lower limit of integration to be 0 (it could have been any value). Applying the initial condition on $u(\xi,\eta)$, we have

$$u(\xi,-\xi) = f(\xi) = \psi(-\xi) + \sqrt{2}\int_0^\xi \sqrt{\frac{s}{s+\xi}}\,g(s)\,ds$$
$$\Longrightarrow \psi(-\xi) = f(\xi) - \sqrt{2}\int_0^\xi \sqrt{\frac{s}{s+\xi}}\,g(s)\,ds$$
$$\Longrightarrow \psi(\eta) = f(-\eta) + \sqrt{2}\int_{-\eta}^0 \sqrt{\frac{s}{s-\eta}}\,g(s)\,ds.$$

Hence, we find that

$$u(\xi,\eta) = f(-\eta) + \sqrt{2}\int_{-\eta}^0 \sqrt{\frac{s}{s-\eta}}\,g(s)\,ds + \sqrt{2}\int_0^\xi \sqrt{\frac{s}{s-\eta}}\,g(s)\,ds$$
$$= f(-\eta) + \sqrt{2}\int_{-\eta}^\xi \sqrt{\frac{s}{s-\eta}}\,g(s)\,ds.$$

This is the solution in terms of the characteristic variables (ξ,η). Substituting in the (x,y) representations for (ξ,η) as determined in Part a implies that the solution in terms of the original variables (x,y), which we denote as $u(x,y)$, is given by

$$u(x,y) = f\left(x^2/2 + x - y\right) + \sqrt{2}\int_{x^2/2+x-y}^{y-x+x^2/2} \sqrt{\frac{s}{s+x^2/2+x-y}}\,g(s)\,ds.$$