Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points

1. Consider the pde
$$u_{tt} + 2u_{xt} - \beta\, u_{xx} = 0,\ x, \beta \in \mathbb{R},\ t > 0.$$
   (a) Determine the stability index Ω as a function of the parameter β.
   (b) Determine the stability of the pde as a function of the parameter β.
   (c) Determine the values of the parameter β for which the Cauchy problem is ill-posed.

2. Consider the linear partial differential operator L defined by
$$Lu = -\nabla \cdot (p \nabla u) + qu, \text{ for } x \in G \subset \mathbb{R}^n,$$
where G is an open, simply-connected bounded region with smooth boundary ∂G, $p = p(x) > 0$ and $q = q(x) \geq 0$, with the boundary condition
$$\alpha(x)\, u + \beta(x) \frac{\partial u}{\partial n} = 0, \text{ where } \alpha, \beta \geq 0,\ \ \alpha + \beta > 0,\ x \in \partial G,$$
and where the inner product is given by
$$(u, w) \equiv \int_G \rho\, u\, w\ dx, \text{ where } \rho = \rho(x) > 0.$$
   (a) Define a self-adjoint operator, and show that $\frac{1}{\rho} L$ is self-adjoint.
   (b) Define a positive operator, and show that $\frac{1}{\rho} L$ is positive.
   (c) Show that the eigenvalues of $\frac{1}{\rho} L$ are non-negative.

3. Let $\mathcal{L}$ be a *positive, self-adjoint, real-valued partial differential operator* defined for all smooth square-integrable functions $h(x)$ where $x \in G \subset \mathbb{R}^n$ that satisfy the boundary condition
$$\alpha h + \beta \frac{\partial h}{\partial n} = 0, \text{ with } \alpha, \beta \geq 0 \text{ where } \alpha + \beta > 0, \text{ for } x \in \partial G.$$
Show that the solution, assuming it exists, to
$$u_{tt} + \mathcal{L}u = F(x,t),\ x \in G,\ t > 0,$$
$$u(x, 0) = f(x),\ \ u_t(x, 0) = g(x) \ \text{ for } x \in G,$$
$$\text{and } \alpha u + \beta \frac{\partial u}{\partial n} = B(x,t) \ \text{ for } x \in \partial G,\ t > 0,$$
is unique. HINT: Assume a unit density function in the inner product.

4. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is an orthonormal sequence of square-integrable functions defined on $x \in G \subset \mathbb{R}^n$ with the inner product

$$(u, w) \equiv \int_G \rho\, u\, w\; dx, \text{ with } \rho = \rho(x) > 0.$$

(a) If $\varphi(x)$ is a square-integrable function for $x \in G$, define the *Fourier Series* for $\varphi(x)$ with respect to $\{\varphi_k(x)\}_{k=1}^{\infty}$.

(b) Beginning with the n^{th} partial sum associated with the Fourier Series for $\varphi(x)$, denoted by $\psi_n(x)$, show that *Bessel's Inequality* holds, i.e.,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2 .$$

(c) Define what it means for a sequence of square-integrable functions $\{\psi_n(x)\}_{n=1}^{\infty}$ to *converge* to a function $\varphi(x)$ *in the mean.*

(d) Show that *Convergence in the Mean* is equivalent to *Parseval's Identity.*