Solutions for Math 436 2019 Final

Question 1: The pde is

$$u_{tt} + 2u_{xt} - \beta\, u_{xx} = 0,$$

Part a: To determine the stability index Ω, we substitute

$$u = a \exp\left(ikx + \lambda t\right) + c.c.,$$

into the the pde to yield

$$\left(\lambda^2 + 2ik\lambda + \beta k^2\right) a \exp\left(ikx + \lambda t\right) + c.c. \;= 0.$$

For non-trivial ($a \neq 0$) solutions

$$\lambda^2 + 2ik\lambda + \beta k^2 = 0$$

$$\Longrightarrow \lambda = -ik \pm \sqrt{-k^2 - \beta k^2} = -ik \pm i\left|k\right| \sqrt{1+\beta}.$$

It therefore follows that

$$\Omega = \text{lub}_k \operatorname{Re}\left[\lambda\left(k\right)\right] = \begin{cases} 0 \text{ if } \beta \geq -1 \\ +\infty \text{ if } \beta < -1. \end{cases}$$

Part b: The pde is *neutrally stable* if $\beta \geq -1$ and is *unstable* if $\beta < -1$.
Part c: The Cauchy problem is ill-posed if $\beta < -1$.
Question 2a: Let $f\left(x\right)$ and $g\left(x\right)$ be smooth square-integrable functions that satisfy the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be self-adjoint if

$$\left(f,\, \mathcal{L}g\right) = \left(g,\, \mathcal{L}f\right).$$

To show $\frac{1}{\rho}L$ is self-adjoint, we will show that

$$\left(f,\, \frac{1}{\rho}Lg\right) - \left(g,\, \frac{1}{\rho}Lf\right) = 0.$$

We have

$$\left(f,\, \frac{1}{\rho}Lg\right) - \left(g,\, \frac{1}{\rho}Lf\right) = \int\limits_G f\, Lg - g\, Lf\;dx$$

$$= \int\limits_G f\,\left[-\nabla\cdot\left(p\nabla g\right) + qg\right] - g\,\left[-\nabla\cdot\left(p\nabla f\right) + qf\right]\;dx$$

$$= \int\limits_G g\,\nabla\cdot\left(p\nabla f\right) - f\,\nabla\cdot\left(p\nabla g\right)\;dx$$

$$= \int\limits_{\partial G} g\,\mathbf{n}\cdot\left(p\nabla f\right) - f\,\mathbf{n}\cdot\left(p\nabla g\right)\;dx + \int\limits_G p\,\left[\nabla f\cdot\nabla g - \nabla g\cdot\nabla f\right]\;dx$$

$$= \int_{\partial G} p \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) \, dx.$$

Now for a given $x \in \partial G$, $\beta = 0$ or $\beta > 0$. If $\beta = 0$ for some $x \in \partial G$, then $f = g = 0$ for those $x \in \partial G$ so that

$$\left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right)_{\text{for those } x \in \partial G} = 0.$$

If $\beta \neq 0$ for some $x \in \partial G$, then for those $x \in \partial G$

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ and } \frac{\partial g}{\partial n} = -\frac{\alpha g}{\beta} \text{ for those } x \in \partial G$$

$$\Longrightarrow \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right)_{\text{for those } x \in \partial G} = \left(\frac{\alpha g f}{\beta} - \frac{\alpha g f}{\beta} \right)_{\text{for those } x \in \partial G} = 0.$$

Thus, regardless of whether β is zero or not for any specific $x \in \partial G$, we have shown

$$\left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right)_{x \in \partial G} = 0 \Longrightarrow \left(f, \, \frac{1}{\rho} L g \right) = \left(g, \, \frac{1}{\rho} L f \right).$$

Question 2b: Let $f(x)$ be a smooth square-integrable function that satisfies the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be positive if

$$(f, \, \mathcal{L} f) \geq 0.$$

To show $\frac{1}{\rho} L$ is positive, we proceed directly:

$$\left(f, \, \frac{1}{\rho} L f \right) = \int_G f \, L f \, dx = \int_G f \, \left[-\nabla \cdot (p \nabla f) + q f \right] \, dx$$

$$= -\int_{\partial G} p f \frac{\partial f}{\partial n} \, dx + \int_G p \, \nabla f \cdot \nabla f + q \, f^2 \, dx.$$

We note that since $p > 0$ and $q \geq 0$

$$\int_G p \, \nabla f \cdot \nabla f + q \, f^2 \, dx \geq 0.$$

Now for a given $x \in \partial G$, $\beta = 0$ or $\beta > 0$. If $\beta = 0$ for some $x \in \partial G$, then $f = 0$ for those $x \in \partial G$

$$\left(p f \frac{\partial f}{\partial n} \right)_{\text{for those } x \in \partial G} = 0.$$

If $\beta \neq 0$ for some $x \in \partial G$, then for those $x \in \partial G$

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ for those } x \in \partial G$$

$$\Longrightarrow \left(pf\frac{\partial f}{\partial n}\right)_{\text{for those } x\in\partial G} = -\left(\frac{\alpha p f^2}{\beta}\right)_{\text{for those } x\in\partial G} \leq 0,$$

since $p > 0$, $q \geq 0$, $\alpha \geq 0$ and $\beta > 0$. Thus, regardless of whether β is zero or not for any specific $x \in \partial G$, we have shown

$$\int_{\partial G} pf\frac{\partial f}{\partial n}\, dx \leq 0.$$

Thus, we have shown

$$\left(f,\, \frac{1}{\rho}Lf\right) = -\int_{\partial G} pf\frac{\partial f}{\partial n}\, dx + \int_{G} p\,\nabla f\cdot\nabla f + q\,f^2\; dx \geq 0.$$

Question 2c: The eigenvalue problem is given by

$$\frac{1}{\rho}Lu = \lambda u,\; x \in G,$$

with the boundary condition

$$\alpha\left(x\right)u + \beta\left(x\right)\frac{\partial u}{\partial n} = 0 \text{ for } x \in \partial G.$$

Since $\frac{1}{\rho}L$ is a positive operator

$$0 \leq \left(u,\, \frac{1}{\rho}Lu\right) = \lambda\left(u,\, u\right) = \lambda\left\|u\right\|^2 \Longrightarrow \lambda \geq 0.$$

Question 3: Assume two solutions exist to the problem denoted by $u_1\left(x,t\right)$ and $u_2\left(x,t\right)$, respectively, i.e.,

$$\partial_{tt}u_1 + \mathcal{L}u_1 = F\left(x,t\right),\; x \in G,\; t > 0,$$

$$u_1\left(x,0\right) = f\left(x\right),\;\; \partial_t u_1\left(x,0\right) = g\left(x\right) \;\text{ for } x \in G,$$

$$\text{and } \alpha u_1 + \beta\frac{\partial u_1}{\partial n} = B\left(x,t\right) \;\text{ for } x \in \partial G,\; t > 0,$$

and

$$\partial_{tt}u_2 + \mathcal{L}u_2 = F\left(x,t\right),\; x \in G,\; t > 0,$$

$$u_2\left(x,0\right) = f\left(x\right),\;\; \partial_t u_2\left(x,0\right) = g\left(x\right) \;\text{ for } x \in G,$$

$$\text{and } \alpha u_2 + \beta\frac{\partial u_2}{\partial n} = B\left(x,t\right) \;\text{ for } x \in \partial G,\; t > 0.$$

Define the difference $w = u_1 - u_2$, it follows that w satisfies

$$w_{tt} + \mathcal{L}w = 0,\; x \in G,\; t > 0,$$

$$w\left(x,0\right) = 0,\;\; w_t\left(x,0\right) = 0 \text{ for } x \in G,$$

$$\text{and } \alpha w + \beta \frac{\partial w}{\partial n} = 0 \text{ for } x \in \partial G,\ t > 0.$$

We form the energy equation

$$w_t w_{tt} + w_t \mathcal{L} w = 0 \Longrightarrow \partial_t \|w_t\|^2 + 2\left(w_t, \mathcal{L}w\right) = 0,$$

and since $\mathcal{L}$ is a *self-adjoint operator* it follows that $(w_t, \mathcal{L}w) = (w, \mathcal{L}w_t)$, so that the energy equation can be written in the form

$$\partial_t \|w_t\|^2 + (w_t, \mathcal{L}w) + (w, \mathcal{L}w_t) = 0 \Longrightarrow \frac{\partial}{\partial t}\left[\|w_t\|^2 + (w, \mathcal{L}w)\right] = 0$$

$$\Longrightarrow \|w_t\|^2 + (w, \mathcal{L}w) = \left[\|w_t\|^2 + (w, \mathcal{L}w)\right]_{t=0} = 0$$

$$\Longrightarrow \|w_t\|^2 = -\left(w, \mathcal{L}w\right) \leq 0,$$

since $\mathcal{L}$ is a *positive operator*, i.e., $(w, \mathcal{L}w) \geq 0$. Hence

$$\|w_t\| = 0 \Longrightarrow w_t = 0 \Longrightarrow w\left(x, t\right) = w\left(x, 0\right) = 0.$$

Question 4a: The Fourier Series is defined as

$$\varphi\left(x\right) = \sum_{k=1}^{\infty} \left(\varphi, \varphi_k\right)\, \varphi_k\left(x\right).$$

Question 4b: The n^{th} partial sum is given by

$$\psi_n\left(x\right) = \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)\, \varphi_k\left(x\right).$$

To show Bessel's Inequality, we begin with

$$0 \leq \|\varphi\left(x\right) - \psi_n\left(x\right)\|^2 = \left(\varphi - \psi_n, \varphi - \psi_n\right) = \left(\varphi, \varphi\right) - 2\left(\varphi, \psi_n\right) + \left(\psi_n, \psi_n\right)$$

$$= \left(\varphi, \varphi\right) - 2\left(\varphi, \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)\, \varphi_k\left(x\right)\right) + \left(\sum_{m=1}^{n} \left(\varphi, \varphi_m\right)\, \varphi_m\left(x\right), \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)\, \varphi_k\left(x\right)\right)$$

$$= \left(\varphi, \varphi\right) - 2\sum_{k=1}^{n} \left(\varphi, \varphi_k\right)\left(\varphi,\ \varphi_k\right) + \sum_{m=1}^{n}\sum_{k=1}^{n} \left(\varphi, \varphi_k\right)\left(\varphi, \varphi_m\right)\left(\varphi_m,\ \varphi_k\right)$$

$$= \left(\varphi, \varphi\right) - 2\sum_{k=1}^{n} \left(\varphi, \varphi_k\right)^2 + \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)^2 = \left(\varphi, \varphi\right) - \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)^2$$

$$\Longrightarrow \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)^2 \leq \left(\varphi, \varphi\right).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit $n \to \infty$, it follows

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi) .$$

Question 4c: Mean square convergence is defined as

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

Question 4d: We must show that

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) .$$

From Question 4b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2 .$$

Thus, provided the limit exists,

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \lim_{n\to\infty} \sum_{k=1}^{n} (\varphi, \varphi_k)^2 .$$

Hence

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0 \Longrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) ,$$

and

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) \Longrightarrow \lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0.$$