Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

30 October 2018

Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. (a) Use the Method of Characteristics to show that the solution $u(x,t)$ to

$$(1+t)\,u_t + x\,u\,u_x = 0,\ -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = f(x),\ -\infty < x < \infty,$$

can be written in the form

$$u(x,t) = f(\tau)\ \text{ with } x = \tau\ (1+t)^{f(\tau)}\ \text{ where } -\infty < \tau < \infty.$$

(b) Show that the *first time of shock formation and its position*, denoted by t_s and x_s, respectively, are determined by

$$t_s = \exp\left[-\frac{1}{\tau_{\min}\, f'(\tau_{\min})}\right] - 1,$$
$$x_s = \tau_{\min} \exp\left[-\frac{f(\tau_{\min})}{\tau_{\min}\, f'(\tau_{\min})}\right],$$

where $\tau_{\min}$ is the minimizer associated with

$$\min_{\tau}\left[-\frac{1}{\tau\ f'(\tau)}\right] \text{ subject to } -\frac{1}{\tau\ f'(\tau)} \geq 0.$$

(c) If $f(\tau) = \exp(-\tau^2)$ show that

$$t_s = \exp(e/2) - 1 \text{ and } x_s = \pm\sqrt{e}.$$

2. Suppose the initial data for the linear $1st$-order pde

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y),$$

is given by

$$u(x,h(x)) = f(x),$$

where $y = h(x)$ is a characteristic. Show that

$$\frac{df}{dx} = \frac{c(x,h)\,f + d(x,h)}{a(x,h)}.$$

3. Show that the solution to the Cauchy problem

$$u_{tt} - u_{xx} = h(x,t),\ -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x),\ -\infty < x < \infty,$$

where $h(x,t)$, $f(x)$ and $g(x)$ are smooth spatially square-integrable functions is unique. HINT: You may assume that $u(x,t)$ and all its derivatives are smooth square-integrable functions with respect to x.

4. (a) Classify the 2^{nd}-order linear pde in $\mathbb{R}^2$

$$x\, u_{xx} + y\, u_{xy} + u_x = 0,$$

as a function of x and y.

(b) In the region(s) where the pde is *hyperbolic* show that the *characteristic variables* (ξ, η) can be written as

$$\xi = y/x \text{ and } \eta = y,$$

and reduce the pde to the *H1 canonical form*

$$u_{\xi\eta} = 0.$$

(c) Integrate the *H1 canonical form* to show that the *general solution to the pde* is given by

$$u(x, y) = \Phi(y/x) + \Psi(y),$$

where Φ and Ψ are arbitrary differentiable functions of their arguments.

(d) Assuming that

$$u\left(x, x^2\right) = f(x) \text{ and } u_y\left(x, x^2\right) = g(x),$$

use the general solution in Part c to show that

$$u(x, y) = 2f(\sqrt{y}) - f(y/x) + 2\int_{\sqrt{y}}^{y/x} s\, g(s)\, ds.$$