Solutions for Math 436 2018 Midterm

Question 1 Part a: The pde and initial condition are given by

$$(1+t)\,u_t + x\,u\,u_x = 0,\ -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x),\ -\infty < x < \infty.$$

The initial data curve, denoted by Γ, can be parametrized as

$$\Gamma = \{(x,t)\ \mid t = 0 \text{ and } x = \tau \text{ with } \tau \in \mathbb{R}\},$$

and the initial data can be parametrically written as

$$u|_{\Gamma} = f(\tau).$$

The characteristic equations can be written as

$$\frac{dt}{ds} = 1 + t \text{ subject to } \left.t\right|_{s=0} = 0 \Longrightarrow \ln(1+t) = s,$$

$$\frac{du}{ds} = 0 \text{ subject to } \left.u\right|_{s=0} = f(\tau) \Longrightarrow u = f(\tau),$$

$$\frac{dx}{ds} = xu = x\,f(\tau) \text{ subject to } \left.x\right|_{s=0} = \tau \Longrightarrow \ln x = s\,f(\tau) + \ln\tau$$

$$\Longrightarrow x = \tau\exp[s\,f(\tau)] = \tau\exp[\ln(1+t)\ f(\tau)] = \tau\ (1+t)^{f(\tau)}.$$

Thus, the solution $u(x,t)$ can be written in the implicit form

$$u(x,t) = f(\tau), \tag{1}$$

where $\tau(x,t)$ is determined by

$$x = \tau\ (1+t)^{f(\tau)} \text{ where } \tau \in \mathbb{R}. \tag{2}$$

Part b. The shock forms the first time $|u_x| \to \infty$. It follows from (1) that

$$u_x = f'(\tau)\,\tau_x$$

and from (2) that

$$1 = \tau_x\ (1+t)^{f(\tau)} + \tau\,(1+t)^{f(\tau)}\ln(1+t)\,f'(\tau)\,\tau_x$$

$$\Longrightarrow \tau_x = \frac{1}{(1+t)^{f(\tau)}\left[1 + \tau\ln(1+t)\,f'(\tau)\right]}.$$

Assuming that there is no "shock" in the initial condition, i.e., $|f'(\tau)| < \infty$, it follows that $|u_x| \to \infty$ only occurs when

$$\ln(1+t) = -\frac{1}{\tau\,f'(\tau)}.$$

Since $\ln(1+t) \geq 0$ for $t \geq 0$, it follows that the first time that a shock will form, denoted by t_s, will be determined by

$$\ln(1+t_s) = \min_{\tau}\left[-\frac{1}{\tau f'(\tau)}\right] \text{ subject to } -\frac{1}{\tau f'(\tau)} \geq 0.$$

If we denote the minimizer with respect to τ as $\tau_{\min}$ we may write

$$\ln(1+t_s) = -\frac{1}{\tau_{\min} f'(\tau_{\min})} \Longrightarrow t_s = \exp\left[-\frac{1}{\tau_{\min} f'(\tau_{\min})}\right] - 1,$$

and thus the position of the first shock to form, denoted by x_s, will be given by

$$x_s = \tau_{\min}(1+t_s)^{f(\tau_{\min})} = \tau_{\min}\exp\left[-\frac{f(\tau_{\min})}{\tau_{\min} f'(\tau_{\min})}\right].$$

Part c. If $f(\tau) = \exp\left(-\tau^2\right)$, then $f'(\tau) = -2\tau\exp\left(-\tau^2\right)$ so that

$$\min_{\tau}\left[-\frac{1}{\tau f'(\tau)}\right] = \min_{\tau}\left[\frac{\exp(\tau^2)}{2\tau^2}\right].$$

The minimizer $\tau_{\min}$ is the solution to

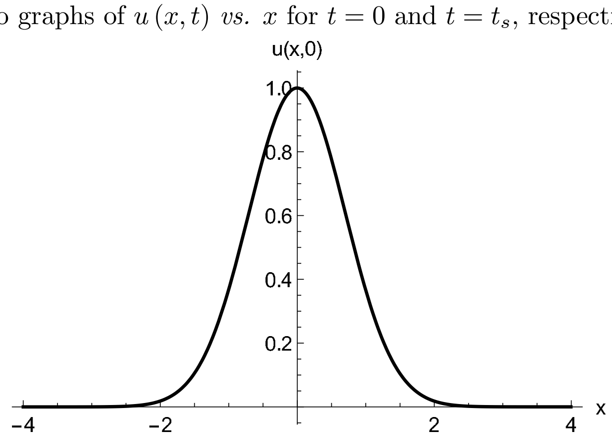
$$\frac{d}{d\tau}\left[\frac{\exp(\tau^2)}{2\tau^2}\right] = \frac{\exp(\tau^2)(\tau^2-1)}{\tau^3} = 0 \Longrightarrow \tau_{\min} = \pm 1.$$

It follows that

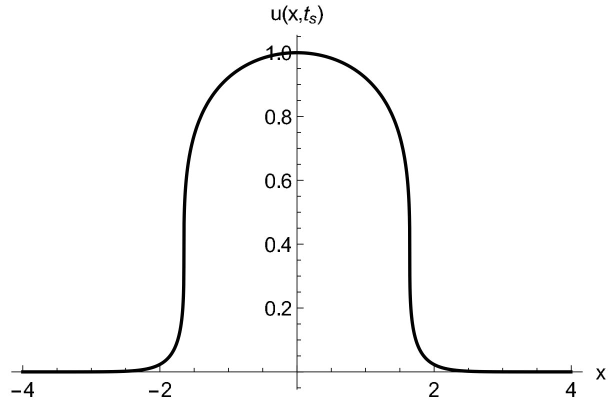
$$t_s = \exp\left[-\frac{1}{\tau_{\min} f'(\tau_{\min})}\right] - 1 = \exp\left[\frac{\exp(\tau_{\min}^2)}{2\tau_{\min}^2}\right] - 1 = e^{e/2} - 1 \simeq 2.89,$$

$$x_s = \tau_{\min}\exp\left[-\frac{f(\tau_{\min})}{\tau_{\min} f'(\tau_{\min})}\right] = \tau_{\min}\exp\left(\frac{1}{2\tau_{\min}^2}\right) = \pm\sqrt{e} \simeq \pm 1.65.$$

Below are two graphs of $u(x,t)$ *vs.* x for $t = 0$ and $t = t_s$, respectively.



$u(x,t)$ vs. x at $t = 0$



$u(x,t)$ vs. x at $t = t_s$

Question 2: The pde and initial condition, is given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)\,,$$

$$u(x,h(x)) = f(x)\,,$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x,h)}{a(x,h)}.$$

It follows that

$$\frac{df}{dx} = u_x(x,h(x)) + u_y(x,h(x))\frac{dh}{dx}$$
$$= u_x(x,h(x)) + \frac{b(x,h)}{a(x,h)}u_y(x,h(x))$$
$$= \frac{a(x,h)\,u_x(x,h(x)) + b(x,h)\,u_y(x,h(x))}{a(x,h)}$$
$$= \frac{c(x,h)\,u(x,h(x)) + d(x,h)}{a(x,h)} = \frac{c(x,h)\,f + d(x,h)}{a(x,h)}.$$

Question 3: The pde and initial conditions are given by

$$u_{tt} - u_{xx} = h(x,t)\,, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x)\,, \quad -\infty < x < \infty,$$

where $h(x,t)$, $f(x)$ and $g(x)$ are smooth spatially square-integrable functions. To show uniqueness, we assume that there are two solutions, given by $u_1(x,t)$ and $u_2(x,t)$, i.e.,

$$(\partial_{tt} - \partial_{xx})\,u_1 = h, \quad -\infty < x < \infty,\ t > 0,$$

$$u_1(x,0) = f \text{ and } \partial_t u_1(x,0) = g, \quad -\infty < x < \infty,$$

and

$$(\partial_{tt} - \partial_{xx})\, u_2 = h, \quad -\infty < x < \infty,\, t > 0,$$

$$u_2\,(x,0) = f \text{ and } \partial_t u_2\,(x,0) = g, \quad -\infty < x < \infty.$$

Let $\Phi\,(x,t) = u_1\,(x,t) - u_2\,(x,t)$. We will show that $\Phi\,(x,t) = 0$ for all $t \geq 0$. Hence $u_1\,(x,t) = u_2\,(x,t)$ and we have established uniqueness. It follows that

$$(\partial_{tt} - \partial_{xx})\, \Phi = 0, \quad -\infty < x < \infty,\, t > 0, \tag{1}$$

$$\Phi\,(x,0) = 0 \text{ and } \Phi_t\,(x,0) = 0, \quad -\infty < x < \infty. \tag{2}$$

The energy equation associated is obtained by multiplying (1) by Φ_t and rewriting the resulting equation as a space-time divergence, i.e.,

$$\Phi_t\,(\partial_{tt} - \partial_{xx})\, \Phi = \frac{1}{2}\partial_t \left[(\Phi_t)^2 + (\Phi_x)^2\right] - \partial_x\,(\Phi_t \Phi_x) = 0.$$

It therefore follows that

$$\partial_t \int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \; dx = 2\Phi_t\Phi_x|_{-\infty}^{\infty} = 0, \tag{3}$$

since $\Phi_t\Phi_x \to 0$ as $|x| \to \infty$ since $\Phi_{x,t}$ are smooth square-integrable functions. Thus, it follows from (3) that

$$\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \; dx = \int_{-\infty}^{\infty} \left[(\Phi_t)^2 + (\Phi_x)^2\right]_{t=0} dx = 0, \tag{4}$$

since $\Phi\,(x,0) = 0$ ($\Longrightarrow \Phi_x\,(x,0) = 0$) and $\Phi_t\,(x,0) = 0$. Further, it then follows from (4) that

$$\Phi_t\,(x,t) = \Phi_x\,(x,t) = 0 \text{ for all } t \geq 0 \Longrightarrow \Phi\,(x,t) = 0 \text{ for all } t \geq 0.$$

Question 4 Part a: The pde is given by

$$x\, u_{xx} + y\, u_{xy} + u_x = 0.$$

There are two methods we can use to classify the pde. If use the "ω-test" we determine the roots of the quadratic

$$x\omega^2 + y\omega + 0 = 0 \Longrightarrow \omega = -y/x \text{ and } 0.$$

If $y \neq 0$ there are two real and distinct solutions for ω and the pde is hyperbolic. Although it appears that there is only one $\omega = 0$ root if $x = 0$, this case is degenerate in the sense that the quadratic for ω reduces to a linear relation. Direct examination of the pde, however, shows that it is hyperbolic if $x = 0$ for $y \neq 0$ (and is in fact already in canonical form). If $y = 0$, we see that we see that there is a single real ω root to the quadratic with multiplicity two. Hence, the pde is parabolic for $y = 0$. At the origin $x = y = 0$ the pde is no longer

second order but is first order and is trivially hyperbolic. Hence, in conclusion the pde is hyperbolic for $y \neq 0$ and parabolic for $y = 0$ for all x.
The second way we can classify this pde, and maybe it is the most straight forward way, is to re-write it in the self-adjoint symmetric form

$$(xu_x)_x + \frac{1}{2}(yu_y)_x + \frac{1}{2}(yu_x)_y - \frac{1}{2}u_x = 0,$$

and then into the matrix form

$$\begin{bmatrix} \partial_x & \partial_y \end{bmatrix} \begin{bmatrix} x & y/2 \\ y/2 & 0 \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} u + \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} u = 0.$$

If we now compute the eigenvalues associated with the real symmetric matrix

$$\begin{bmatrix} x & y/2 \\ y/2 & 0 \end{bmatrix} \Longrightarrow \begin{vmatrix} x - \lambda & y/2 \\ y/2 & -\lambda \end{vmatrix} = 0$$

$$\Longrightarrow \lambda^2 - x\lambda - \frac{y^2}{4} = 0 \Longrightarrow \lambda = \frac{x \pm \sqrt{x^2 + y^2}}{2},$$

from which we see that if $y \neq 0$ that necessarily one λ root is positive and the other negative for all x, implying the pde is hyperbolic in this case. However, if $y = 0$, then at least one λ is zero so the pde is parabolic. In the case where $x = y = 0$ this test is degenerate in that the real symmetric matrix is the null matrix.
Part b. Based on the roots to the ω-quadratic in Part a, the characteristic variables are the solutions of

$$\left(\frac{dy}{dx}\right)_\xi = \frac{y}{x} \Longrightarrow \xi = \frac{y}{x} \text{ and } \left(\frac{dy}{dx}\right)_\eta = 0 \Longrightarrow \eta = y.$$

To reduce to *H1 canonical form* we introduce the formal change of variable $(x, y) \longrightarrow (\xi, \eta)$, which leads to the transformations

$$u_x = u_\xi \xi_x + u_\eta \eta_x = -\frac{y}{x^2}u_\xi,$$

$$u_{xx} = 2\frac{y}{x^3}u_\xi + \frac{y^2}{x^4}u_{\xi\xi}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = \frac{1}{x}u_\xi + u_\eta,$$

$$u_{xy} = -\left(\frac{y}{x^2}u_\xi\right)_y = -\frac{1}{x^2}u_\xi - \frac{y}{x^2}\left(\frac{1}{x}u_{\xi\xi} + u_{\xi\eta}\right),$$

which when substituted into the pde yields

$$2\frac{y}{x^2}u_\xi + \frac{y^2}{x^3}u_{\xi\xi} - \frac{y}{x^2}u_\xi - \frac{y^2}{x^3}u_{\xi\xi} - \frac{y^2}{x^2}u_{\xi\eta} - \frac{y}{x^2}u_\xi = 0$$

$$\Longleftrightarrow u_{\xi\eta} = 0. \tag{1}$$

This is the *H1 canonical form*.

Part c. If we integrate (1) with respect to η we obtain

$$u_\xi = \phi(\xi), \tag{2}$$

where $\phi(\xi)$ is an arbitrary functions of its argument. And if we integrate (2) with respect to ξ we obtain

$$u = \Phi(\xi) + \Psi(\eta), \tag{3}$$

where Φ and Ψ are arbitrary functions of their arguments. Equation (3) is the general solution to (1), which in turn implies that the general solution to the original pde can be written in the form

$$u(x, y) = \Phi(y/x) + \Psi(y). \tag{4}$$

Part d. Assuming the initial condition

$$u\left(x, x^2\right) = f(x) \text{ and } u_y\left(x, x^2\right) = g(x),$$

it follows from (4) that

$$\Phi(x) + \Psi\left(x^2\right) = f(x), \tag{5}$$

$$\frac{1}{x}\Phi'(x) + \Psi'\left(x^2\right) = g(x), \tag{6}$$

respectively, where "prime" means differentiation with respect to the argument. It follows from (5) that

$$\Phi'(x) + 2x\,\Psi'\left(x^2\right) = f'(x),$$

which when combined with (6) implies

$$\Phi'(x) = -f'(x) + 2xg(x) \Longrightarrow \Phi(x) = -f(x) + 2\int_0^x s\,g(s)\,ds, \tag{7}$$

where, without loss of generality, we have set the lower limit of integration to zero in the integral. It follows from (5) and (7) that

$$\Psi\left(x^2\right) = f(x) - \Phi(x) = 2f(x) - 2\int_0^x s\,g(s)\,ds$$

$$\Longrightarrow \Psi(x) = 2f\left(\sqrt{x}\right) + 2\int_{\sqrt{x}}^0 s\,g(s)\,ds.$$

Hence we have

$$\begin{aligned} u(x, y) &= \Phi(y/x) + \Psi(y) \\ &= -f(y/x) + 2\int_0^{y/x} s\,g(s)\,ds + 2f(\sqrt{y}) + 2\int_{\sqrt{y}}^0 s\,g(s)\,ds \\ &= 2f(\sqrt{y}) - f(y/x) + 2\int_{\sqrt{y}}^{y/x} s\,g(s)\,ds. \end{aligned}$$