Solutions for Math 436 2018 Final

Question 1a: If $y = h(x)$ are the characteristics, then $h(x)$ must satisfy

$$|I - h'(x)A| = \begin{vmatrix} 1-h' & -2h' \\ -2h' & 1-h' \end{vmatrix} = (1-h')^2 - 4(h')^2 = 0$$

$$\Longrightarrow 1 - h' = \pm 2h' \Longrightarrow h' = \frac{1}{3} \text{ or } -1.$$

Since all of the h' are real and distinct, *the system is strictly or totally hyperbolic.*
Question 1b: To reduce to canonical form we first need to compute the right eigenvectors associated with h' computed in part a. The *right eigenvectors* of A associated with the eigenvalues $1/h'$, i.e., -1 and 3, and denoted by $\mathbf{r}_{-1}$ and $\mathbf{r}_3$, respectively, are determined by

$$(I + A)\,\mathbf{r}_{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{r}_{-1} = \mathbf{0} \Longrightarrow \mathbf{r}_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\left(I - \tfrac{1}{3}A\right)\mathbf{r}_3 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \mathbf{r}_3 = \mathbf{0} \Longrightarrow \mathbf{r}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that $\mathbf{r}_{-1} \cdot \mathbf{r}_3 = 0$. Thus, introducing the transformation

$$\mathbf{u} \equiv R\mathbf{v} = R\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \text{ where } R \equiv \begin{bmatrix} \mathbf{r}_{-1} & \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Longrightarrow R^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

into the pde, leads to

$$\mathbf{v}_y + D\mathbf{v}_x = \mathbf{0}, \text{ where } D \equiv R^{-1}AR = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix},$$

or, in component form,

$$\partial_y v_1 - \partial_x v_1 = 0 \text{ and } \partial_y v_2 + 3\partial_x v_2 = 0.$$

Question 2: The pde is

$$u_t - u_{xx} - u_x + au = 0.$$

To compute the stability index we assume a plane wave solution in the form

$$u = A\exp(ikx + \lambda t) + c.c..$$

Substitution into the pde yields

$$\lambda = -k^2 - a + ik \Longrightarrow \mathrm{Re}(\lambda) = -k^2 - a.$$

Thus,

$$\Omega = lub_k \left[\mathrm{Re} \left(\lambda \right) \right] = -a.$$

Hence, if $a > 0$, the pde is *strictly stable*, if $a = 0$, the pde is *neutrally stable* and if $a < 0$, the pde is *unstable*.

Question 3a: Let $f(x)$ and $g(x)$ be smooth square-integrable functions that satisfy the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be self-adjoint if

$$(f, \, \mathcal{L}g) = (g, \, \mathcal{L}f) \, .$$

To show $\frac{1}{\rho}L$ is self-adjoint, we will show that

$$\left(f, \, \frac{1}{\rho} Lg \right) - \left(g, \, \frac{1}{\rho} Lf \right) = 0.$$

We have

$$\left(f, \, \frac{1}{\rho} Lg \right) - \left(g, \, \frac{1}{\rho} Lf \right) = \int\limits_G f \, Lg - g \, Lf \; dx$$

$$= \int\limits_G f \, \left[-\nabla \cdot (p \nabla g) + qg \right] - g \, \left[-\nabla \cdot (p \nabla f) + qf \right] \; dx$$

$$= \int\limits_G g \, \nabla \cdot (p \nabla f) - f \, \nabla \cdot (p \nabla g) \; dx$$

$$= \int\limits_{\partial G} g \, \mathbf{n} \cdot (p \nabla f) - f \, \mathbf{n} \cdot (p \nabla g) \; dx + \int\limits_G p \, \left[\nabla f \cdot \nabla g - \nabla g \cdot \nabla f \right] \; dx$$

$$= \int\limits_{\partial G} p \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) \, dx.$$

If $\beta = 0$, then $f = g = 0$ for $x \in \partial G$ so that

$$\left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right)_{\partial G} = 0.$$

If $\beta \neq 0$, then

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ and } \frac{\partial g}{\partial n} = -\frac{\alpha g}{\beta} \text{ for } x \in \partial G$$

$$\Longrightarrow \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right)_{\partial G} = \left(\frac{\alpha g f}{\beta} - \frac{\alpha g f}{\beta} \right)_{\partial G} = 0.$$

Thus, regardless of whether β is zero or not for $x \in \partial G$, we have shown

$$\left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right)_{\partial G} = 0 \Longrightarrow \left(f, \, \frac{1}{\rho} Lg \right) = \left(g, \, \frac{1}{\rho} Lf \right).$$

Question 3b: Let $f(x)$ be a smooth square-integrable function that satisfies the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be positive if

$$(f,\, \mathcal{L}f) \geq 0.$$

To show $\frac{1}{\rho}L$ is positive, we proceed directly:

$$\left(f,\, \frac{1}{\rho}Lf\right) = \int\limits_G f\, Lf\ dx = \int\limits_G f\ \left[-\nabla \cdot (p\nabla f) + qf\right]\ dx$$

$$= -\int\limits_{\partial G} pf\frac{\partial f}{\partial n}\, dx + \int\limits_G p\, \nabla f \cdot \nabla f + q\, f^2\ dx.$$

If $\beta = 0$, then $f = 0$ for $x \in \partial G$ so that

$$\left(f,\, \frac{1}{\rho}Lf\right) = \int\limits_G p\, \nabla f \cdot \nabla f + q\, f^2\ dx \geq 0,$$

since $p > 0$ and $q \geq 0$. If $\beta \neq 0$, then

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ for } x \in \partial G$$

$$\Longrightarrow \left(f,\, \frac{1}{\rho}Lf\right) = \int\limits_{\partial G} \frac{\alpha p f^2}{\beta}\, dx + \int\limits_G p\, \nabla f \cdot \nabla f + q\, f^2\ dx \geq 0,$$

since $p > 0$, $q \geq 0$, $\alpha \geq 0$ and $\beta > 0$.

Question 3c: The eigenvalue problem is given by

$$\frac{1}{\rho}Lu = \lambda u,\ x \in G,$$

with the boundary condition

$$\alpha(x)\, u + \beta(x)\, \frac{\partial u}{\partial n} = 0 \text{ for } x \in \partial G.$$

Since $\frac{1}{\rho}L$ is a positive operator

$$0 \leq \left(u,\, \frac{1}{\rho}Lu\right) = \lambda\,(u,\, u) = \lambda\, \|u\|^2 \Longrightarrow \lambda \geq 0.$$

Question 4a: The Fourier Series is defined as

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k)\ \varphi_k(x).$$

Question 4b: The n^{th} partial sum is given by

$$\psi_n(x) = \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x).$$

To show Bessel's Inequality, we begin with

$$0 \le \|\varphi(x) - \psi_n(x)\|^2 = (\varphi - \psi_n, \varphi - \psi_n) = (\varphi, \varphi) - 2(\varphi, \psi_n) + (\psi_n, \psi_n)$$

$$= (\varphi, \varphi) - 2\left(\varphi, \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right) + \left(\sum_{m=1}^{n} (\varphi, \varphi_m)\ \varphi_m(x), \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_k) + \sum_{m=1}^{n}\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_m)(\varphi_m, \varphi_k)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)^2 + \sum_{k=1}^{n} (\varphi, \varphi_k)^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2$$

$$\Longrightarrow \sum_{k=1}^{n} (\varphi, \varphi_k)^2 \le (\varphi, \varphi).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit $n \to \infty$, it follows

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \le (\varphi, \varphi).$$

Question 4c: Mean square convergence is defined as

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

Question 4d: We must show that

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi).$$

From Question 4b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Thus, provided the limit exists,

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \lim_{n\to\infty} \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Hence

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0 \Longrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi),$$

and

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) \Longrightarrow \lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0.$$