Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Consider the 1^{st}-order pde
$$\sin(x)\, u_x + y\cos(x)\, u_y = -\cos(x)\, u^2. \tag{1}$$
   (a) Show that the characteristic curves associated with (1) are given by
   $$\xi = \frac{\sin x}{y},$$
   for constant ξ.

   (b) By introducing an appropriate change of independent variables, show that the general solution to (1) can be written in the form
   $$u(x,y) = \frac{1}{\ln(\sin x) + \phi\left(\frac{\sin x}{y}\right)}, \tag{2}$$
   where ϕ is an arbitrary function of its argument.

   (c) Assuming the initial condition
   $$u(x,1) = \csc x,$$
   use (2) to show that
   $$u(x,y) = \frac{y}{y\ln y + \sin x}.$$

2. Suppose that the initial data for the linear 1^{st}-order pde
$$a(x,y)\, u_x + b(x,y)\, u_y = c(x,y)\, u + d(x,y), \tag{3}$$
is given by
$$u(x, h(x)) = f(x),$$
where $y = h(x)$ is a characteristic associated with (3), and a, b, c and d are all smooth functions. Show that it is impossible to uniquely determine *both* $u_x(x,h(x))$ and $u_y(x,h(x))$, if a solution exists at all.

3. Consider the quasi-linear 1^{st}-order pde
$$u_t + u^2 u_x = 0 \text{ for } -\infty < x < \infty,\ t > 0,$$
with the initial condition
$$u(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

(a) Use the *Method of Characteristics* to show that the solution can be written in the form

$$u = f(\tau),$$

with

$$x = t\,[f(\tau)]^2 + \tau,$$

for $-\infty < \tau < \infty$.

(b) Show that the *first time of shock formation*, denoted by t_s, is given by

$$t_s = \min_{\tau}\left[-\frac{1}{2f'(\tau)\,f(\tau)}\right] \text{ where } t_s \geq 0,$$

and $f'(\tau) = df(\tau)/d\tau$.

(c) For

$$f(x) = \begin{cases} 0 \text{ for } x \leq -1, \\ 1 + x \text{ for } -1 < x \leq 0, \\ 1 - x \text{ for } 0 < x \leq 1, \\ 0 \text{ for } x > 1, \end{cases}$$

use the results of Parts a and b to show that

$$t_s = \frac{1}{2} \text{ and } x_s = \frac{1}{2},$$

where x_s is the x coordinate of the first shock to form.

4. Consider the linear pde in $\mathbb{R}^2$ given by

$$u_{xx} - 2x\,u_{xy} - (1 + 2x)\,u_{yy} = 0. \tag{4}$$

(a) Show that (4) is *hyperbolic* if $x \neq -1$ and *parabolic* if $x = -1$.

(b) Assuming that $x \neq -1$, show that the characteristic variables can be written as

$$\xi = y + x + x^2 \text{ and } \eta = x - y.$$

HINT: If η is a characteristic variable so is $-\eta$.

(c) Assuming that $x \neq -1$, show that the $H1$ canonical form of (4) can be written in the form

$$u_{\xi\,\eta} + \frac{1}{2(1 + \xi + \eta)}u_{\xi} = 0. \tag{5}$$

(d) Starting from (5) show that the general solution to (4) can be written in the form

$$u(x, y) = \psi(x - y) + \int^{y+x+x^2} \frac{\phi(s)}{\sqrt{1 + x - y + s}}\,ds,$$

where ψ and ϕ are arbitrary functions of their arguments. HINT: Introduce an integrating factor into (5).