Solutions for Math 436 2017 Midterm

Question 1: The pde is given by

$$\sin(x)\,u_x + y\cos(x)\,u_y = -\cos(x)\,u^2.$$

Part (a). The characteristics are the level curves associated with the solution to the ode

$$\frac{dy}{dx} = \frac{y\cos x}{\sin x} \Longrightarrow \int \frac{dy}{y} = \int \frac{\cos x}{\sin x}dx$$

$$\Longrightarrow \ln y + \ln \xi = \ln(\sin x) \Longrightarrow \xi = \frac{\sin x}{y},$$

for constant ξ.

Part (b). To find the general solution we transform from (x, y) to (ξ, η) variables where ξ is the characteristic variable and η is any other independent variable, say,

$$\xi = \frac{\sin x}{y} \text{ and } \eta = x.$$

It follows that

$$u_x = u_\xi \xi_x + u_\eta \eta_x = \frac{\cos x}{y}u_\xi + u_\eta,$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = -\frac{\sin x}{y^2}u_\xi.$$

Substitution into the pde yields

$$\frac{\sin(x)\cos(x)}{y}u_\xi + \sin(x)\,u_\eta - \frac{\sin(x)\cos(x)}{y}u_\xi = \sin(x)\,u_\eta = -\cos(x)\,u^2$$

$$\Longleftrightarrow u_\eta = -\frac{\cos\eta}{\sin\eta}u^2 \Longrightarrow -\int \frac{du}{u^2} = \int \frac{\cos\eta}{\sin\eta}d\eta \Longrightarrow \frac{1}{u} = \ln(\sin\eta) + \phi(\xi)$$

$$\Longleftrightarrow u = \frac{1}{\ln(\sin\eta) + \phi(\xi)} \Longrightarrow u(x,y) = \frac{1}{\ln(\sin x) + \phi\left(\frac{\sin x}{y}\right)}.$$

where $\phi(\xi)$ is an arbitrary function of its argument.

Part (c). If $u(x, 1) = \csc x$, it follows from the general solution that

$$\csc x = \frac{1}{\ln(\sin x) + \phi(\sin x)} \Longrightarrow \sin x = \ln(\sin x) + \phi(\sin x)$$

$$\Longrightarrow \phi(*) = * - \ln(*),$$

and thus

$$u(x,y) = \frac{1}{\ln(\sin x) + \frac{\sin x}{y} - \ln\left(\frac{\sin x}{y}\right)} = \frac{1}{\frac{\sin x}{y} + \ln y} = \frac{y}{y\ln y + \sin x}.$$

Question 2: The pde and initial condition are given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)\,,$$

$$u(x,h(x)) = f(x)\,,$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x,h)}{a(x,h)}.$$

Differentiating the initial condition with respect to x leads to

$$\frac{df}{dx} = u_x(x,h(x)) + u_y(x,h(x))\,\frac{dh}{dx},$$

but we also have, from the pde itself, that on $y = h(x)$

$$a(x,h)\,u_x(x,h(x)) + b(x,h)\,u_y(x,h(x)) = c(x,h)\,f(x) + d(x,h)\,.$$

These two equations can be written as the 2×2 system

$$\left[\begin{array}{cc} 1 & dh/dx \\ a(x,h) & b(x,h) \end{array}\right]\left[\begin{array}{c} u_x(x,h(x)) \\ u_y(x,h(x)) \end{array}\right] = \left[\begin{array}{c} df/dx \\ c(x,h)\,f(x) + d(x,h) \end{array}\right].$$

If $u_x(x,h(x))$ and $u_y(x,h(x))$ were *uniquely determined*, assuming a solution exists at all, it would imply that the coefficient matrix

$$\left[\begin{array}{cc} 1 & dh/dx \\ a(x,h) & b(x,h) \end{array}\right],$$

was invertible. But

$$\left|\begin{array}{cc} 1 & dh/dx \\ a(x,h) & b(x,h) \end{array}\right| = b(x,h) - a(x,h)\,\frac{dh}{dx} = 0,$$

so the inverse of the coefficient matrix doesn't exist implying that $u_x(x,h(x))$ and $u_y(x,h(x))$ *cannot be uniquely determined*.

Question 3: The pde and (general) initial condition are given by

$$u_t + u^2 u_x = 0 \text{ for } -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

Part (a). The initial data curve may be written parametrically as

$$x = \tau \text{ with } \tau \in \mathbb{R}, \text{ and } t = 0.$$

Thus, the characteristic equations can be written as

$$\frac{du}{ds} = 0 \text{ with } u|_{s=0} = f(\tau)\,,$$

$$\frac{dt}{ds} = 1 \text{ with } t|_{s=0} = 0,$$

$$\frac{dx}{ds} = u^2 \text{ with } x|_{s=0} = \tau,$$

where s determines the parametric dependence along the characteristics. The characteristic equations for u and t can be immediately integrated with respect to s to yield

$$u = f(\tau) \text{ and } t = s,$$

which if substituted into the characteristic equation for x implies

$$\frac{dx}{ds} = [f(\tau)]^2 \text{ with } x|_{s=0} = \tau,$$

and since s and τ are independent variables, this can be integrated with respect to s to yield

$$x = s[f(\tau)]^2 + \tau = t\,[f(\tau)]^2 + \tau.$$

Thus, the solution for $u(x,t)$ can be written in the implicit coupled form

$$u = f(\tau),$$

$$x = t\,[f(\tau)]^2 + \tau.$$

Part (b). A shock will form when $|u_x| \to \infty$. From the solution we compute

$$u_x = f'(\tau)\,\tau_x,$$

$$1 = [2\,t\,f(\tau)\;f'(\tau) + 1]\,\tau_x \Longrightarrow \tau_x = \frac{1}{1 + 2\,t\,f(\tau)\;f'(\tau)}$$

$$\Longrightarrow u_x = \frac{f'(\tau)}{1 + 2\,t\,f(\tau)\;f'(\tau)},$$

from which we conclude that

$$|u_x| \to \infty \text{ for } t = -\frac{1}{2\;f(\tau)\;f'(\tau)},$$

so that the *first time that a shock can form*, denoted as t_s, will be given by

$$t_s = \min_{\tau}\left[-\frac{1}{2\;f(\tau)\;f'(\tau)}\right] \text{ where } t_s \geq 0.$$

Part (c). For

$$f(\tau) = \begin{cases} 0 \text{ for } \tau \leq -1, \\ 1 + \tau \text{ for } -1 < \tau \leq 0, \\ 1 - \tau \text{ for } 0 < \tau \leq 1, \\ 0 \text{ for } \tau > 1, \end{cases}$$

we have

$$f'(\tau) = \begin{cases} 0 \text{ for } \tau < -1, \\ 1 \text{ for } -1 < \tau < 0, \\ -1 \text{ for } 0 < \tau < 1, \\ 0 \text{ for } \tau > 1. \end{cases}$$

We see that $f(\tau) \geq 0$ so in order to have $t_s \geq 0$ we will require $f'(\tau) < 0$. But $f'(\tau) < 0$ only for $0 < \tau < 1$ where $f(\tau) = 1 - \tau$. Hence

$$t_s = \min_{0 < \tau < 1} \left[\frac{1}{2(1-\tau)} \right] \Longrightarrow \tau_{\min} = 0^+,$$

where $\tau_{\min}$ is the minimizer, which in turn implies

$$t_s = \frac{1}{2} \Longrightarrow x_s = t_s \left[f(\tau_{\min}) \right]^2 + \tau_{\min} = \frac{1}{2}.$$

Question 4: The pde is given by

$$u_{xx} - 2x\, u_{xy} - (1+2x)\, u_{yy} = 0.$$

Part (a). To classify we consider the roots of

$$\omega^2 - 2x\omega - (1+2x) = 0$$

$$\Longrightarrow \omega^{+,-} = \frac{2x \pm \sqrt{4x^2 + 4(1+2x)}}{2} = x \pm \sqrt{(1+x)^2}$$
$$= x \pm |1+x| = x \pm (1+x),$$

without loss of generality. There are two real distinct roots if $x \neq -1$ and one double real root if $x = -1$. Hence the pde is *hyperbolic* if $x \neq -1$ and *parabolic* if $x = -1$.

Part (b). In the hyperbolic case where $x \neq -1$, the characteristic variables, denoted by ξ and η, are determined by, respectively,

$$\left(\frac{dy}{dx} \right)_\xi = -\omega^+ = -1 - 2x \Longrightarrow y = -x - x^2 + \xi \Longrightarrow \xi = y + x + x^2,$$

$$\left(\frac{dy}{dx} \right)_\eta = -\omega^- = 1 \Longrightarrow y = x - \eta \Longrightarrow \eta = x - y,$$

where, as it turns out, it is convenient to write the integration constant η with a minus sign.

Part (c). To derive the $H1$ canonical form of the pde in the hyperbolic case where $x \neq -1$, we introduce the characteristic variables

$$\xi = y + x + x^2 \text{ and } \eta = x - y.$$

It follows that

$$u_x = u_\xi \xi_x + u_\eta \eta_x = (1+2x)\, u_\xi + u_\eta,$$

$$
\begin{aligned}
u_{xx} &= (1+2x)\, u_{\xi x} + u_{\eta x} + 2u_{\xi} \\
&= (1+2x)\left[(1+2x)\, u_{\xi\xi} + u_{\eta\xi}\right] + (1+2x)\, u_{\xi\,\eta} + u_{\eta\eta} + 2u_{\xi} \\
&= (1+2x)^2\, u_{\xi\xi} + 2\,(1+2x)\, u_{\xi\,\eta} + u_{\eta\eta} + 2u_{\xi}, \\
u_y &= u_{\xi}\xi_y + u_{\eta}\eta_y = u_{\xi} - u_{\eta}, \\
u_{yy} &= u_{\xi\xi} - 2u_{\xi\,\eta} + u_{\eta\eta}, \\
u_{xy} &= u_{\xi x} - u_{\eta x} = (1+2x)\, u_{\xi\xi} + u_{\xi\,\eta} - (1+2x)\, u_{\xi\,\eta} - u_{\eta\eta} \\
&= (1+2x)\, u_{\xi\xi} - 2x u_{\xi\,\eta} - u_{\eta\eta}.
\end{aligned}
$$

Substitution into the pde yields

$$
\begin{gathered}
(1+2x)^2\, u_{\xi\xi} + 2\,(1+2x)\, u_{\xi\,\eta} + u_{\eta\eta} + 2u_{\xi} \\
-2x\left[(1+2x)\, u_{\xi\xi} - 2x u_{\xi\,\eta} - u_{\eta\eta}\right] \\
-(1+2x)\left[u_{\xi\xi} - 2u_{\xi\,\eta} + u_{\eta\eta}\right] = 0,
\end{gathered}
$$

which simplifies to

$$
\begin{gathered}
4\left(1+2x+x^2\right) u_{\xi\,\eta} + 2u_{\xi} = 0 \\
\Longrightarrow u_{\xi\,\eta} + \frac{1}{2\,(1+\xi+\eta)} u_{\xi} = 0,
\end{gathered}
$$

since $x^2 + 2x = \xi + \eta$. This is the $H1$ canonical form of the pde.

Part (d). It is possible to introduce an integrating factor with respect to η differentiation that allows one to obtain a general solution. The integrating factor is given by

$$
\exp\left[\frac{1}{2}\int \frac{d\eta}{1+\xi+\eta}\right] = \exp\left[\frac{1}{2}\ln(1+\xi+\eta)\right] = \exp\ln\sqrt{1+\xi+\eta} = \sqrt{1+\xi+\eta}.
$$

Multiplication of the $H1$ canonical form through by the integrating factor leads to

$$
\begin{gathered}
\sqrt{1+\xi+\eta}\, u_{\xi\,\eta} + \frac{1}{2\sqrt{1+\xi+\eta}}\, u_{\xi} = 0 \\
\Longrightarrow \left(\sqrt{1+\xi+\eta}\, u_{\xi}\right)_{\eta} = 0 \\
\Longrightarrow \sqrt{1+\xi+\eta}\, u_{\xi} = \phi(\xi) \\
\Longrightarrow u = \psi(\eta) + \int^{\xi} \frac{\phi(s)}{\sqrt{1+\eta+s}}\, ds,
\end{gathered}
$$

where ψ and ϕ are arbitrary functions of their arguments. Hence it follows that

$$
u(x,y) = \psi(x-y) + \int^{y+x+x^2} \frac{\phi(s)}{\sqrt{1+x-y+s}}\, ds.
$$