Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Consider the pde

$$u_{tt} + 2u_{xt} - \beta\, u_{xx} + u = 0,\ x, \beta \in \mathbb{R},\ t > 0.$$

   (a) Determine the stability index Ω as a function of the parameter β.

   (b) Determine the stability of the pde as a function of the parameter β.

   (c) Determine the values of the parameter β for which the Cauchy problem is ill-posed.

2. The linear shallow water equations for a rotating fluid can be written in the form

$$u_t - v = -h_x, \tag{1}$$

$$v_t + u = -h_y, \tag{2}$$

$$h_t + u_x + v_y = 0, \tag{3}$$

where (1) and (2) are the x and y *momentum equations*, respectively, (3) is the *mass equation*, u and v are the x and y *velocities*, respectively, and $h \geq 0$ is the fluid *depth*.

   (a) Show that (1) and (2) can be combined together to yield

$$(\partial_{tt} + 1)\, u = -h_y - h_{xt},$$

$$(\partial_{tt} + 1)\, v = h_x - h_{yt}.$$

   (b) Use Part (a) to show that (3) can be written in the form

$$(\partial_{tt} + 1 - \partial_{xx} - \partial_{yy})\, h_t = 0.$$

   (c) Consider the "channel" domain

$$G = \{(x, y) \mid -\infty < x < \infty,\ 0 < y < 1\},$$

   with the Dirichlet boundary conditions $h(x, 0, t) = h(x, 1, t) = 0$. Assuming the *neutrally-stable* along-channel propagating normal mode solution

$$h = a \sin(\pi y) \exp(ikx - i\omega t) + c.c.,$$

   where a is the amplitude, k is the real x-direction wavenumber, and ω is the real frequency, use Part (b) to show that the *dispersion relation* (for $\omega \neq 0$) is given by

$$\omega = \pm\sqrt{1 + \pi^2 + k^2}.$$

   (d) Show that the $\omega \neq 0$ normal modes are *dispersive*.

3. Consider the linear partial differential operator L defined by

$$Lu = -\nabla \cdot (p\nabla u) + qu, \text{ for } x \in G \subset \mathbb{R}^n,$$

where G is an open, simply-connected bounded region with smooth boundary ∂G, $p = p(x) > 0$ and $q = q(x) \geq 0$, with the boundary condition

$$\alpha(x) u + \beta(x) \frac{\partial u}{\partial n} = 0, \text{ where } \alpha, \beta \geq 0,\ \alpha + \beta > 0,\ x \in \partial G,$$

and where the inner product is given by

$$(u, w) \equiv \int_G \rho\, u\, w\, dx, \text{ where } \rho = \rho(x) > 0.$$

(a) Define a self-adjoint operator, and show that $\frac{1}{\rho}L$ is self-adjoint.

(b) Define a positive operator, and show that $\frac{1}{\rho}L$ is positive.

(c) Show that the eigenvalues of $\frac{1}{\rho}L$ are non-negative.

4. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is an orthonormal sequence of square-integrable functions defined on $x \in G \subset \mathbb{R}^n$ with the inner product

$$(u, w) \equiv \int_G \rho\, u\, w\, dx, \text{ with } \rho = \rho(x) > 0.$$

(a) If $\varphi(x)$ is a square-integrable function for $x \in G$, define the *Fourier Series* for $\varphi(x)$ with respect to $\{\varphi_k(x)\}_{k=1}^{\infty}$.

(b) Beginning with the n^{th} partial sum associated with the Fourier Series for $\varphi(x)$, denoted by $\psi_n(x)$, show that *Bessel's Inequality* holds, i.e.,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2.$$

(c) Define what it means for a sequence of square-integrable functions $\{\psi_n(x)\}_{n=1}^{\infty}$ to *converge* to a function $\varphi(x)$ *in the mean.*

(d) Show that *convergence in the mean* is equivalent to *Parseval's Identity.*