Solutions for Math 436 2017 Final

1. The pde is

$$u_{tt} + 2u_{xt} - \beta\, u_{xx} + u = 0,$$

   (a) To determine the stability index Ω, we substitute

$$u = a \exp(ikx + \lambda t) + c.c.,$$

   into the the pde to yield

$$\left(\lambda^2 + 2ik\lambda + \beta k^2 + 1\right) a \exp(ikx + \lambda t) + c.c. \ = 0.$$

   For non-trivial ($a \neq 0$) solutions

$$\lambda^2 + 2ik\lambda + \beta k^2 + 1 = 0$$

$$\Longrightarrow \lambda = -ik \pm \sqrt{-1 - (1+\beta)\, k^2} = -ik \pm i\sqrt{1 + (1+\beta)\, k^2}.$$

   It therefore follows that

$$\Omega = \mathrm{lub}_k\, \mathrm{Re}\left[\lambda(k)\right] = \begin{cases} 0 \text{ if } \beta \geq -1 \\ +\infty \text{ if } \beta < -1. \end{cases}$$

   (b) The pde is *neutrally stable* if $\beta \geq -1$ and is *unstable* if $\beta < -1$.

   (c) The Cauchy problem is ill-posed if $\beta < -1$.

2. The linear shallow water equations are given by

$$u_t - v = -h_x, \tag{1}$$

$$v_t + u = -h_y, \tag{2}$$

$$h_t + u_x + v_y = 0, \tag{3}$$

   (a) It follows from (1) and (2) that

$$u_{tt} - v_t = -h_{xt},$$

$$v_{tt} + u_t = -h_{yt},$$

   which if u_t and v_t are eliminated in these using (1) and (2) again results in

$$u_{tt} - (-u - h_y) = -h_{xt},$$

$$v_{tt} + (v - h_x) = -h_{yt},$$

   which simplifies to

$$(\partial_{tt} + 1)\, u = -h_y - h_{xt}, \tag{4}$$

$$(\partial_{tt} + 1)\, v = h_x - h_{yt}. \tag{5}$$

(b) It follows from (3) that

$$(\partial_{tt}+1)\,h_t + [(\partial_{tt}+1)\,u]_x + [(\partial_{tt}+1)\,v]_y = 0,$$

which if we substitute in the result from Part (a) implies that

$$(\partial_{tt}+1)\,h_t - h_{xy} - h_{xxt} + h_{xy} - h_{yyt} = 0,$$

which simplifies to

$$(\partial_{tt}+1-\partial_{xx}-\partial_{yy})\,h_t = 0. \qquad (6)$$

(c) Substitution of the *neutrally-stable* along-channel propagating normal mode solution into (6)

$$h = a\sin(\pi y)\exp(ikx - i\omega t) + c.c.,$$

leads to

$$-i\omega\left(-\omega^2+1+k^2+\pi^2\right)a\sin(\pi y)\exp(ikx-i\omega t) + c.c. \;= 0.$$

For a non-trivial solution it follows that

$$\omega\,\left(-\omega^2+1+k^2+\pi^2\right) = 0,$$

which implies that for the $\omega \neq 0$ solutions

$$\omega = \pm\sqrt{1+\pi^2+k^2}.$$

(d) The *phase velocity* c is given by

$$c = \frac{\omega}{k} = \pm\frac{\sqrt{1+\pi^2+k^2}}{k} \Longrightarrow \frac{dc}{dk} = \mp\frac{1+\pi^2}{k^2\sqrt{1+\pi^2+k^2}} \neq 0.$$

(a) Let $f(x)$ and $g(x)$ be smooth square-integrable functions that satisfy the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be self-adjoint if

$$(f,\,\mathcal{L}g) = (g,\,\mathcal{L}f)\,.$$

To show $\frac{1}{\rho}L$ is self-adjoint, we will show that

$$\left(f,\,\frac{1}{\rho}Lg\right) - \left(g,\,\frac{1}{\rho}Lf\right) = 0.$$

We have

$$\left(f,\,\frac{1}{\rho}Lg\right) - \left(g,\,\frac{1}{\rho}Lf\right) = \int_G f\,Lg - g\,Lf\;dx$$

$$= \int_G f\,\left[-\nabla\cdot(p\nabla g)+qg\right] - g\,\left[-\nabla\cdot(p\nabla f)+qf\right]\;dx$$

$$= \int_G g\,\nabla\cdot(p\nabla f) - f\,\nabla\cdot(p\nabla g)\;dx$$

$$= \int_{\partial G} g\,\mathbf{n}\cdot(p\nabla f) - f\,\mathbf{n}\cdot(p\nabla g)\;dx + \int_G p\,\left[\nabla f\cdot\nabla g - \nabla g\cdot\nabla f\right]\;dx$$

$$= \int_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\;dx.$$

If $\beta = 0$, then $f = g = 0$ for $x \in \partial G$ so that

$$\int_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\;dx = 0.$$

If $\beta \neq 0$, then

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ and } \frac{\partial g}{\partial n} = -\frac{\alpha g}{\beta} \text{ for } x \in \partial G$$

$$\implies \int_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\;dx = \int_{\partial G} p\left(\frac{\alpha g f}{\beta} - \frac{\alpha g f}{\beta}\right)\;dx = 0.$$

Thus we have shown that

$$\left(f,\,\frac{1}{\rho}Lg\right) = \left(g,\,\frac{1}{\rho}Lf\right).$$

(b) Let $f\,(x)$ be a smooth square-integrable function that satisfies the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be positive if

$$(f,\,\mathcal{L}f) \geq 0.$$

To show $\frac{1}{\rho}L$ is positive, we proceed directly:

$$\left(f,\,\frac{1}{\rho}Lf\right) = \int_G f\,Lf\;dx = \int_G f\,\left[-\nabla\cdot(p\nabla f)+qf\right]\;dx$$

$$= -\int_{\partial G} pf\frac{\partial f}{\partial n}\,dx + \int_G p\,\nabla f\cdot\nabla f + q\,f^2\;dx.$$

If $\beta = 0$, then $f = 0$ for $x \in \partial G$ so that

$$\left(f,\,\frac{1}{\rho}Lf\right) = \int_G p\,\nabla f\cdot\nabla f + q\,f^2\;dx \geq 0,$$

since $p > 0$ and $q \geq 0$. If $\beta \neq 0$, then

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ for } x \in \partial G$$

$$\Longrightarrow \left(f, \, \frac{1}{\rho} L f \right) = \int\limits_{\partial G} \frac{\alpha p f^2}{\beta} \, dx + \int\limits_{G} p \, \nabla f \cdot \nabla f + q \, f^2 \, dx \geq 0,$$

since $p > 0$, $q \geq 0$, $\alpha \geq 0$ and $\beta > 0$.

(c) The eigenvalue problem is given by

$$\frac{1}{\rho} L u = \lambda u, \, x \in G,$$

with the boundary condition

$$\alpha \left(x \right) u + \beta \left(x \right) \frac{\partial u}{\partial n} = 0 \text{ for } x \in \partial G.$$

Since $\frac{1}{\rho} L$ is a positive operator

$$0 \leq \left(u, \, \frac{1}{\rho} L u \right) = \lambda \left(u, \, u \right) = \lambda \left\| u \right\|^2 \Longrightarrow \lambda \geq 0.$$

(a) The Fourier Series is defined as

$$\varphi \left(x \right) = \sum_{k=1}^{\infty} \left(\varphi, \varphi_k \right) \, \varphi_k \left(x \right).$$

(b) The n^{th} partial sum is given by

$$\psi_n \left(x \right) = \sum_{k=1}^{n} \left(\varphi, \varphi_k \right) \, \varphi_k \left(x \right).$$

To show Bessel's Inequality, we begin with

$$0 \leq \left\| \varphi \left(x \right) - \psi_n \left(x \right) \right\|^2 = \left(\varphi - \psi_n, \varphi - \psi_n \right) = \left(\varphi, \varphi \right) - 2 \left(\varphi, \psi_n \right) + \left(\psi_n, \psi_n \right)$$

$$= \left(\varphi, \varphi \right) - 2 \left(\varphi, \sum_{k=1}^{n} \left(\varphi, \varphi_k \right) \, \varphi_k \left(x \right) \right) + \left(\sum_{m=1}^{n} \left(\varphi, \varphi_m \right) \, \varphi_m \left(x \right), \sum_{k=1}^{n} \left(\varphi, \varphi_k \right) \, \varphi_k \left(x \right) \right)$$

$$= \left(\varphi, \varphi \right) - 2 \sum_{k=1}^{n} \left(\varphi, \varphi_k \right) \left(\varphi, \, \varphi_k \right) + \sum_{m=1}^{n} \sum_{k=1}^{n} \left(\varphi, \varphi_k \right) \left(\varphi, \varphi_m \right) \left(\varphi_m, \, \varphi_k \right)$$

$$= \left(\varphi, \varphi \right) - 2 \sum_{k=1}^{n} \left(\varphi, \varphi_k \right)^2 + \sum_{k=1}^{n} \left(\varphi, \varphi_k \right)^2 = \left(\varphi, \varphi \right) - \sum_{k=1}^{n} \left(\varphi, \varphi_k \right)^2$$

$$\Longrightarrow \sum_{k=1}^{n} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit $n \to \infty$, it follows

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).$$

(c) Mean square convergence is defined as

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

(d) We must show that

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi).$$

From Question 4b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Thus, provided the limit exists,

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \lim_{n\to\infty} \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Hence

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0 \Longrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi),$$

and

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) \Longrightarrow \lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0.$$