Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. The *linear* shallow water equations can be written as the 2×2 system of partial differential equations

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = \mathbf{0},$$

where $g > 0$ and $H > 0$ are positive constants.

   (a) *Classify* the 2×2 system.

   (b) *Reduce* the 2×2 system to *canonical form*, which is given by

$$\left(\frac{h}{\sqrt{H}} + \frac{u}{\sqrt{g}}\right)_t + \sqrt{gH}\left(\frac{h}{\sqrt{H}} + \frac{u}{\sqrt{g}}\right)_x = 0,$$

$$\left(\frac{h}{\sqrt{H}} - \frac{u}{\sqrt{g}}\right)_t - \sqrt{gH}\left(\frac{h}{\sqrt{H}} - \frac{u}{\sqrt{g}}\right)_x = 0.$$

   (c) Using the *Method of Characteristics*, solve the canonical equations for $h(x,t)$ and $u(x,t)$ assuming the initial conditions $h(x,0) = f(x)$ and $u(x,0) = 0$, thereby showing that

$$h(x,t) = \frac{1}{2}\left[f\left(x - \sqrt{gH}t\right) + f\left(x + \sqrt{gH}t\right)\right],$$

$$u(x,t) = \frac{1}{2}\sqrt{\frac{g}{H}}\left[f\left(x - \sqrt{gH}t\right) - f\left(x + \sqrt{gH}t\right)\right].$$

2. Consider the $1st$-order linear partial differential equation

$$xu_x + yu_y = u.$$

   (a) Determine the *general solution* for $u(x,y)$.

   (b) Using the general solution, find the solution for $u(x,y)$ if $u(x,x^2) = \sin x$.

3. Suppose the initial data for the linear $1st$-order partial differential equation

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y),$$

is given by

$$u(x,h(x)) = f(x),$$

where $y = h(x)$ is a characteristic. Show that

$$\frac{df}{dx} = \frac{c(x,h)\,f + d(x,h)}{a(x,h)}.$$

4. Consider the scalar linear partial differential equation in $\mathbb{R}^2$ given by

$$u_{xx} + 2u_{xy} + u_{yy} + u_x + u_y = 0.$$

(a) *Classify* and *reduce to canonical form.*

(b) Starting from the canonical form, find the general solution for $u(x, y)$.