Solutions for Math 436 2016 Midterm

Question 1: The *linear* shallow water equations are given by

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = \mathbf{0}, \tag{1}$$

where $g > 0$ and $H > 0$ are positive constants.
(a) To classify we compute ω from

$$\left| I - \omega \begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix} \right| = \begin{vmatrix} 1 & -\omega H \\ -\omega g & 1 \end{vmatrix} = 0 \Longrightarrow \omega = \pm \frac{1}{\sqrt{gH}}.$$

Since the ω are real and distinct, the system is strictly or totally hyperbolic.
(b) To reduce to canonical form we need the right "eigenvector" associated with each ω. For $\omega = 1/\sqrt{gH}$, we have

$$\begin{pmatrix} 1 & -\sqrt{\frac{H}{g}} \\ -\sqrt{\frac{g}{H}} & 1 \end{pmatrix} \mathbf{r}_1 = \mathbf{0} \Longrightarrow \mathbf{r}_1 = \begin{pmatrix} \sqrt{H} \\ \sqrt{g} \end{pmatrix},$$

and for $\omega = -1/\sqrt{gH}$, we have

$$\begin{pmatrix} 1 & \sqrt{\frac{H}{g}} \\ \sqrt{\frac{g}{H}} & 1 \end{pmatrix} \mathbf{r}_2 = \mathbf{0} \Longrightarrow \mathbf{r}_2 = \begin{pmatrix} \sqrt{H} \\ -\sqrt{g} \end{pmatrix}.$$

Define

$$R = \begin{pmatrix} \sqrt{H} & \sqrt{H} \\ \sqrt{g} & -\sqrt{g} \end{pmatrix} \Longrightarrow R^{-1} = \frac{1}{2} \begin{pmatrix} 1/\sqrt{H} & 1/\sqrt{g} \\ 1/\sqrt{H} & -1/\sqrt{g} \end{pmatrix}.$$

Introducing the change of dependent variable

$$\mathbf{u} = R\mathbf{v} \Longleftrightarrow \mathbf{v} = R^{-1} \begin{pmatrix} h \\ u \end{pmatrix} = \frac{1}{2} \begin{pmatrix} h/\sqrt{H} + u/\sqrt{g} \\ h/\sqrt{H} - u/\sqrt{g} \end{pmatrix},$$

into (1) leads to

$$\mathbf{v}_t + R^{-1} \begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix} R\, \mathbf{v}_x = \mathbf{0},$$

or, equivalently,

$$\mathbf{v}_t + \frac{1}{2} \begin{pmatrix} 1/\sqrt{H} & 1/\sqrt{g} \\ 1/\sqrt{H} & -1/\sqrt{g} \end{pmatrix} \begin{pmatrix} 0 & H \\ g & 0 \end{pmatrix} \begin{pmatrix} \sqrt{H} & \sqrt{H} \\ \sqrt{g} & -\sqrt{g} \end{pmatrix} \mathbf{v}_x = \mathbf{0},$$

or, equivalently,

$$\begin{pmatrix} h/\sqrt{H} + u/\sqrt{g} \\ h/\sqrt{H} - u/\sqrt{g} \end{pmatrix}_t + \begin{pmatrix} \sqrt{gH} & 0 \\ 0 & -\sqrt{gH} \end{pmatrix} \begin{pmatrix} h/\sqrt{H} + u/\sqrt{g} \\ h/\sqrt{H} - u/\sqrt{g} \end{pmatrix}_x = \mathbf{0},$$

which can be written in the form

$$\left(\frac{h}{\sqrt{H}}+\frac{u}{\sqrt{g}}\right)_t+\sqrt{gH}\left(\frac{h}{\sqrt{H}}+\frac{u}{\sqrt{g}}\right)_x=0, \tag{2}$$

$$\left(\frac{h}{\sqrt{H}}-\frac{u}{\sqrt{g}}\right)_t-\sqrt{gH}\left(\frac{h}{\sqrt{H}}-\frac{u}{\sqrt{g}}\right)_x=0. \tag{3}$$

(c) We assume that $u(x,0)=0$ and $h(x,0)=f(x)$. The initial data curve can be parameterized as $t=0$ and $x=\tau$. The characteristic equations for (2) are therefore

$$\frac{d}{ds}\left(\frac{h}{\sqrt{H}}+\frac{u}{\sqrt{g}}\right)=0, \text{ with } \left.\left(\frac{h}{\sqrt{H}}+\frac{u}{\sqrt{g}}\right)\right|_{s=0}=\frac{f(\tau)}{\sqrt{H}},$$

$$\frac{dx}{ds}=\sqrt{gH}, \text{ with } \left.x\right|_{s=0}=\tau \text{ and } \frac{dt}{ds}=1, \text{ with } \left.t\right|_{s=0}=0,$$

which can be solved to yield

$$\tau=x-\sqrt{gH}t \text{ and } s=t,$$

$$\frac{h}{\sqrt{H}}+\frac{u}{\sqrt{g}}=\frac{f(\tau)}{\sqrt{H}},$$

and thus that

$$h(x,t)+\sqrt{\frac{H}{g}}u(x,t)=f\left(x-\sqrt{gH}t\right).$$

Similarly, it follows from (3) that

$$h(x,t)-\sqrt{\frac{H}{g}}u(x,t)=f\left(x+\sqrt{gH}t\right),$$

which implies that

$$h(x,t)=\frac{1}{2}\left[f\left(x-\sqrt{gH}t\right)+f\left(x+\sqrt{gH}t\right)\right],$$

$$u(x,t)=\frac{1}{2}\sqrt{\frac{g}{H}}\left[f\left(x-\sqrt{gH}t\right)-f\left(x+\sqrt{gH}t\right)\right].$$

Question 2: The pde is given by

$$xu_x+yu_y=u.$$

(a) The characteristics are given by the level curves associated with

$$\frac{dy}{dx}=\frac{y}{x}\Longrightarrow\xi=y/x.$$

Let us introduce the new variables

$$\xi = y/x \text{ and } \eta = x \text{ (or any other independent variable).}$$

We have

$$\partial_x = \xi_x \partial_\xi + \eta_x \partial_\eta = -\frac{y}{x^2}\partial_\xi + \partial_\eta,$$

$$\partial_y = \xi_y \partial_\xi + \eta_y \partial_\eta = \frac{1}{x}\partial_\xi,$$

so that substitution into the pde yields

$$x\left(-\frac{y}{x^2}u_\xi + u_\eta\right) + \frac{y}{x}u_\xi = \eta u_\eta = u,$$

implying that

$$u = A(\xi)\,\eta \Longrightarrow u(x,y) = xA(y/x), \tag{4}$$

where $A(\xi)$ is an arbitrary function of its argument. Equation (4) gives the general solution.
(b) If $u\left(x, x^2\right) = \sin x$, it follows from (4) that

$$xA(x) = \sin x \Longrightarrow A(x) = \frac{\sin x}{x} \Longrightarrow u(x,y) = x\frac{\sin(y/x)}{y/x} = \frac{x^2 \sin(y/x)}{y}.$$

Question 3: The pde and initial condition is given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y),$$

$$u(x, h(x)) = f(x),$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x,h)}{a(x,h)}.$$

It follows that

$$\frac{df}{dx} = u_x(x, h(x)) + u_y(x, h(x))\frac{dh}{dx}$$

$$= u_x(x,h(x)) + \frac{b(x,h)}{a(x,h)}u_y(x,h(x)) = \frac{a(x,h)\,u_x(x,h(x)) + b(x,h)\,u_y(x,h(x))}{a(x,h)}$$

$$= \frac{c(x,h)\,u(x,h(x)) + d(x,h)}{a(x,h)} = \frac{c(x,h)\,f + d(x,h)}{a(x,h)}.$$

Question 4: The pde is given by

$$u_{xx} + 2u_{xy} + u_{yy} + u_x + u_y = 0.$$

Since the problem is posed in $\mathbb{R}^2$, we will use the theory developed in class specifically for linear pdes in $\mathbb{R}^2$ as opposed to the more general theory developed

for scalar pdes in $\mathbb{R}^n$, although we note that the later approach works and yields, of course, the same result.
(a) To classify the pde we introduce the $\omega(x,y)$ functions as determined by

$$\omega^2 + 2\omega + 1 = 0 \Longleftrightarrow (\omega+1)^2 = 0 \Longleftrightarrow \omega = -1.$$

Since $\omega = -1$ is double root, the pde is *parabolic*.

To transform the pde into canonical form we first introduce the characteristic variable ξ, as determined from

$$\left(\frac{dy}{dx}\right)_\xi = -\omega = 1 \Longrightarrow \xi = y - x,$$

and introduce the additional independent variable $\eta = x$ (or any other independent variable). Thus, introducing the coordinate transformation $(x,y) \to (\xi,\eta)$ leads to

$$\partial_x = \xi_x \partial_\xi + \eta_x \partial_\eta = -\partial_\xi + \partial_\eta \Longrightarrow \partial_{xx} = \partial_{\xi\xi} - 2\partial_{\xi\eta} + \partial_{\eta\eta},$$

$$\partial_y = \xi_y \partial_\xi + \eta_y \partial_\eta = \partial_\xi \Longrightarrow \partial_{yy} = \partial_{\xi\xi},$$

$$\partial_{xy} = -\partial_{\xi\xi} + \partial_{\xi\eta},$$

where we assume the mixed partials are equal, which, when substituted into the pde, yields

$$(\partial_{\xi\xi} - 2\partial_{\xi\eta} + \partial_{\eta\eta})\, u + 2\left(-\partial_{\xi\xi} + \partial_{\eta\xi}\right) u + u_{\xi\xi} + \left(-\partial_\xi + \partial_\eta\right) u + u_\xi = 0$$

which simplifies to

$$u_{\eta\eta} + u_\eta = 0. \tag{5}$$

Equation (5) is the canonical form of the pde.
(b) Equation (5) can be integrated once to yield

$$u_\eta + u = A(\xi), \tag{(6)}$$

where $A(\xi)$ is an arbitrary function of its argument. Equation (6) can be written in the form

$$\frac{d\left(e^\eta u\right)}{d\eta} = e^\eta A(\xi),$$

which can be integrated again to yield

$$u = A(\xi) + B(\xi)\, e^{-\eta},$$

where $B(\xi)$ is an arbitrary function of its argument. Substituting in for (ξ,η) in terms of (x,y) therefore yields the general solution

$$u(x,y) = A(y-x) + B(y-x)\, e^{-x}.$$