Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

13 December 2016

Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Determine the stability index Ω for the pde

$$u_t - u_{xx} - u_x + au = 0,$$

as a function of the parameter a. Hence, determine the stability of the pde as a function of the parameter a.

2. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is an orthonormal sequence of square-integrable functions defined on $x \in G \subset \mathbb{R}^n$ with the inner product

$$(u, w) \equiv \int_G \rho\, u\, w \, dx, \text{ with } \rho = \rho(x) > 0.$$

   (a) If $\varphi(x)$ is a square-integrable function for $x \in G$, define the *Fourier Series* for $\varphi(x)$ with respect to $\{\varphi_k(x)\}_{k=1}^{\infty}$.

   (b) Beginning with the n^{th} partial sum associated with the Fourier Series for $\varphi(x)$, denoted by $\psi_n(x)$, show that *Bessel's Inequality* holds, i.e.,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2 .$$

   (c) Define what it means for a sequence of square-integrable functions $\{\psi_n(x)\}_{n=1}^{\infty}$ to *converge* to a function $\varphi(x)$ *in the mean.*

   (d) Show that *convergence in the mean* is equivalent to *Parseval's Identity.*

3. Let $\mathcal{L}$ be a *positive, self-adjoint, real-valued partial differential operator* defined for smooth square-integrable functions $f(x)$ where $x \in G \subset \mathbb{R}^n$ and satisfying the boundary conditions

$$\alpha f + \beta \frac{\partial f}{\partial n} = 0, \text{ with } \alpha, \beta \geq 0 \text{ where } \alpha + \beta > 0, \text{ for } x \in \partial G.$$

Show that the solution, assuming it exists, to

$$u_{tt} + \mathcal{L}u = F(x,t), \; x \in G, \; t > 0,$$

$$u(x,0) = f(x), \;\; u_t(x,0) = g(x) \;\text{ for } x \in G,$$

$$\text{and } \alpha u + \beta \frac{\partial u}{\partial n} = B(x,t) \;\text{ for } x \in \partial G, \; t > 0,$$

is unique. HINT: Assume a unit density function in the inner product.

4. Consider the wave equation in spherical coordinates written in the form

$$u_{tt} - c^2 \left[u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left(u_{\phi\phi} + \cot(\phi) u_\phi + \csc^2(\phi) u_{\theta\theta} \right) \right] = 0,$$

where $0 \leq r < 1$, $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$ and $t > 0$.

(a) Assuming $u = v(r,t)\cos(\phi)$, show that

$$v_{tt} - c^2 \left(v_{rr} + \frac{2}{r} v_r - \frac{2}{r^2} v \right) = 0. \tag{1}$$

The radial eigenfunctions associated with (1) are the solutions to the *spherical Bessel equation of order one*, which in self-adjoint form, is given by

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr} \right) + \left(\lambda^2 r^2 - 2 \right) R = 0. \tag{2}$$

The solution to (2) that is bounded at $r = 0$ is the *spherical Bessel function of the first kind of order one,* given by

$$R(r) = j_1(\lambda r).$$

(b) Show that

$$R(1) = j_1(\lambda) = 0 \Longleftrightarrow \tan(\lambda) = \lambda.$$

Let the countable infinity of positive solutions to this relation be denoted by $\{\lambda_n\}_{n=1}^{\infty}$ where $0 < \lambda_1 < \lambda_2 \cdots$.

(c) Show that

$$\left(j_1(\lambda_n r), j_1(\lambda_m r) \right) = \int_0^1 j_1(\lambda_n r) j_1(\lambda_m r) r^2 \, dr = 0 \text{ if } n \neq m.$$

(d) Show that

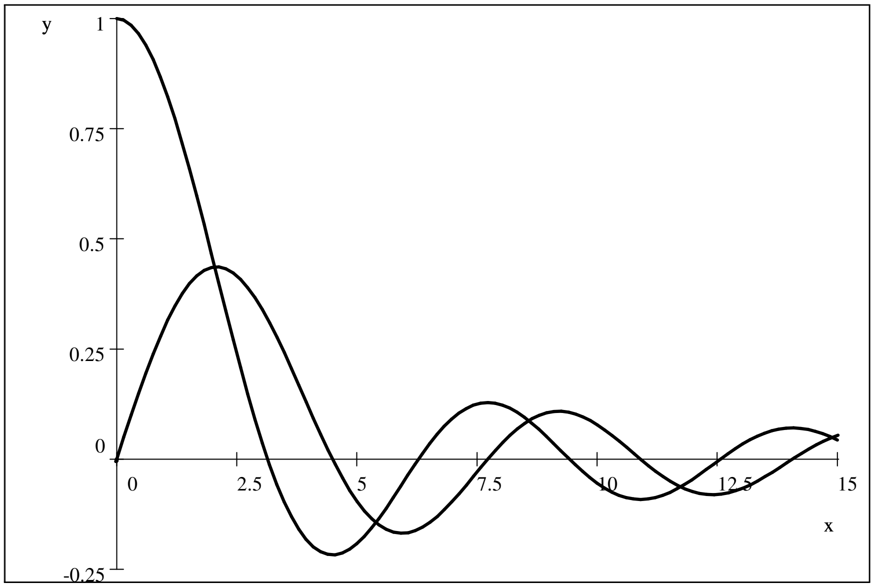
$$\int_0^1 j_1^2(\lambda_n r) r^2 \, dr = \tfrac{1}{2} j_0^2(\lambda_n).$$

Useful Formulae Sheet

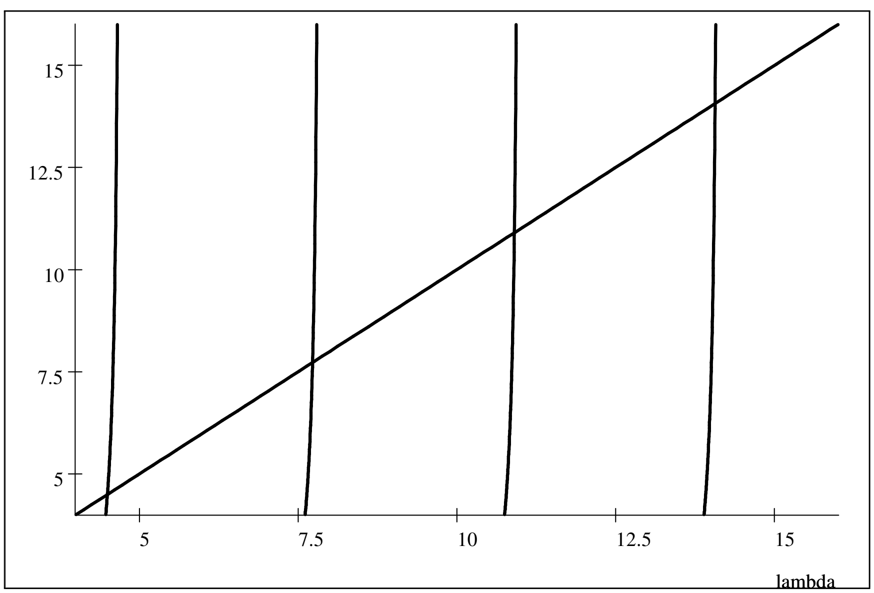
Let $j_n(x)$ be the *spherical Bessel function of the first kind of order* n, then

$$j_0(x) = \frac{\sin(x)}{x} \text{ and } j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x},$$

$$j_1(x) = -\frac{d}{dx} j_0(x), \quad \lim_{x \to 0} j_0(x) = 1 \text{ and } \lim_{x \to 0} j_1(x) = 0,$$



Plot of $j_0(x)$ and $j_1(x)$.



Plot of $\tan(\lambda)$ and λ *vs.* λ. The positive intersection points are the solutions of $\tan(\lambda) = \lambda$.