Solutions for Math 436 2016 Final

Question 1: The pde is

$$u_t - u_{xx} - u_x + au = 0.$$

To compute the stability index we assume a plane wave solution in the form

$$u = A\exp(ikx + \lambda t) + c.c..$$

Substitution into the pde yields

$$\lambda = -k^2 - a + ik \Longrightarrow \mathrm{Re}(\lambda) = -k^2 - a.$$

Thus,

$$\Omega = lub_k\left[\mathrm{Re}(\lambda)\right] = -a.$$

Hence, if $a > 0$, the pde is *strictly stable*, if $a = 0$, the pde is *neutrally stable* and if $a < 0$, the pde is *unstable*.

Question 2a: The Fourier Series is defined as

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k)\ \varphi_k(x).$$

Question 2b: The n^{th} partial sum is given by

$$\psi_n(x) = \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x).$$

To show Bessel's Inequality, we begin with

$$0 \le \|\varphi(x) - \psi_n(x)\|^2 = (\varphi - \psi_n, \varphi - \psi_n) = (\varphi, \varphi) - 2(\varphi, \psi_n) + (\psi_n, \psi_n)$$

$$= (\varphi, \varphi) - 2\left(\varphi, \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right) + \left(\sum_{m=1}^{n} (\varphi, \varphi_m)\ \varphi_m(x), \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_k) + \sum_{m=1}^{n}\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_m)(\varphi_m, \varphi_k)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)^2 + \sum_{k=1}^{n} (\varphi, \varphi_k)^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2$$

$$\Longrightarrow \sum_{k=1}^{n} (\varphi, \varphi_k)^2 \le (\varphi, \varphi).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit $n \to \infty$, it follows

$$\sum_{k=1}^{\infty} \left(\varphi, \varphi_k\right)^2 \leq \left(\varphi, \varphi\right).$$

Question 2c: Mean square convergence is defined as

$$\lim_{n\to\infty} \left\|\varphi\left(x\right) - \psi_n\left(x\right)\right\| = 0.$$

Question 2d: We must show that

$$\lim_{n\to\infty} \left\|\varphi\left(x\right) - \psi_n\left(x\right)\right\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} \left(\varphi, \varphi_k\right)^2 = \left(\varphi, \varphi\right).$$

From Question 2b, we have

$$\left\|\varphi\left(x\right) - \psi_n\left(x\right)\right\|^2 = \left(\varphi, \varphi\right) - \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)^2.$$

Thus, provided the limit exists,

$$\lim_{n\to\infty} \left\|\varphi\left(x\right) - \psi_n\left(x\right)\right\|^2 = \left(\varphi, \varphi\right) - \lim_{n\to\infty} \sum_{k=1}^{n} \left(\varphi, \varphi_k\right)^2.$$

Hence

$$\lim_{n\to\infty} \left\|\varphi\left(x\right) - \psi_n\left(x\right)\right\|^2 = 0 \Longrightarrow \sum_{k=1}^{\infty} \left(\varphi, \varphi_k\right)^2 = \left(\varphi, \varphi\right),$$

and

$$\sum_{k=1}^{\infty} \left(\varphi, \varphi_k\right)^2 = \left(\varphi, \varphi\right) \Longrightarrow \lim_{n\to\infty} \left\|\varphi\left(x\right) - \psi_n\left(x\right)\right\|^2 = 0.$$

Question 3: Assume two solutions exist to the problem denoted by $u_1\left(x,t\right)$ and $u_2\left(x,t\right)$, respectively, i.e.,

$$\partial_{tt} u_1 + \mathcal{L} u_1 = F\left(x,t\right),\ x \in G,\ t > 0,$$

$$u_1\left(x,0\right) = f\left(x\right),\ \ \partial_t u_1\left(x,0\right) = g\left(x\right) \ \text{ for } x \in G,$$

$$\text{and } \alpha u_1 + \beta \frac{\partial u_1}{\partial n} = B\left(x,t\right) \ \text{ for } x \in \partial G,\ t > 0,$$

and

$$\partial_{tt} u_2 + \mathcal{L} u_2 = F\left(x,t\right),\ x \in G,\ t > 0,$$

$$u_2\left(x,0\right) = f\left(x\right),\ \ \partial_t u_2\left(x,0\right) = g\left(x\right) \ \text{ for } x \in G,$$

$$\text{and } \alpha u_2 + \beta \frac{\partial u_2}{\partial n} = B\left(x,t\right) \ \text{ for } x \in \partial G,\ t > 0.$$

Define the difference $w = u_1 - u_2$, it follows that w satisfies

$$w_{tt} + \mathcal{L}w = 0,\ x \in G,\ t > 0,$$

$$w(x, 0) = 0,\ \ w_t(x, 0) = 0 \text{ for } x \in G,$$

$$\text{and } \alpha w + \beta \frac{\partial w}{\partial n} = 0 \text{ for } x \in \partial G,\ t > 0.$$

We form the energy equation

$$w_t w_{tt} + w_t \mathcal{L}w = 0 \Longrightarrow \partial_t \left\| w_t \right\|^2 + 2\left(w_t, \mathcal{L}w\right) = 0,$$

and since $\mathcal{L}$ is a *self-adjoint operator* it follows that $(w_t, \mathcal{L}w) = (w, \mathcal{L}w_t)$, so that the energy equation can be written in the form

$$\partial_t \left\| w_t \right\|^2 + (w_t, \mathcal{L}w) + (w, \mathcal{L}w_t) = 0 \Longrightarrow \frac{\partial}{\partial t}\left[\left\| w_t \right\|^2 + (w, \mathcal{L}w)\right] = 0$$

$$\Longrightarrow \left\| w_t \right\|^2 + (w, \mathcal{L}w) = \left[\left\| w_t \right\|^2 + (w, \mathcal{L}w)\right]_{t=0} = 0$$

$$\Longrightarrow \left\| w_t \right\|^2 = -\left(w, \mathcal{L}w\right) \leq 0,$$

since $\mathcal{L}$ is a *positive operator*, i.e., $(w, \mathcal{L}w) \geq 0$. Hence

$$\left\| w_t \right\| = 0 \Longrightarrow w_t = 0 \Longrightarrow w(x, t) = w(x, 0) = 0.$$

Question 3a: The result follows from direct substitution upon noting that

$$u_{\phi\phi} + \cot(\phi)\, u_\phi = -2\cos(\phi)\, v(r, t)\,.$$

Question 3b: From the Useful Formulae sheet, we have

$$j_1(\lambda) = 0 = \frac{\sin(\lambda)}{\lambda^2} - \frac{\cos(\lambda)}{\lambda} \Longrightarrow \tan(\lambda) = \lambda.$$

Question 3c: To show the orthogonality relationship we begin with the pair of equations

$$\frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_n r)}{dr}\right) + \left(\lambda_n^2 r^2 - 2\right) j_1(\lambda_n r) = 0, \tag{1}$$

$$\frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_m r)}{dr}\right) + \left(\lambda_m^2 r^2 - 2\right) j_1(\lambda_m r) = 0. \tag{2}$$

Multiplying (1) by $j_1(\lambda_m r)$ and (2) $j_1(\lambda_n r)$ and subtracting, we get

$$\left(\lambda_m^2 - \lambda_n^2\right) r^2 j_1(\lambda_n r)\, j_1(\lambda_m r)$$

$$= j_1(\lambda_m r)\frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_n r)}{dr}\right) - j_1(\lambda_n r)\frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_m r)}{dr}\right),$$

which if we integrate with respect to r over the interval $(0,1)$, yields

$$\left(\lambda_m^2 - \lambda_n^2\right) \int_0^1 r^2 j_1(\lambda_n r)\, j_1(\lambda_m r)\; dr$$

$$= \left[r^2 \left(j_1(\lambda_m r) \frac{dj_1(\lambda_n r)}{dr} - j_1(\lambda_n r) \frac{dj_1(\lambda_m r)}{dr} \right) \right]_0^1 = 0$$

$$\Longrightarrow \int_0^1 r^2 j_1(\lambda_n r)\, j_1(\lambda_m r)\; dr = 0 \text{ if } \lambda_m \neq \lambda_n, \text{ i.e., } n \neq m.$$

Question 3d: To show this relationship, we use the formula

$$j_1(x) = -\frac{d}{dx} j_0(x).$$

So that

$$\int_0^1 r^2 j_1^2(\lambda_n r)\; dr = -\frac{1}{\lambda_n} \int_0^1 r^2 j_1(\lambda_n r) \frac{d}{dr} j_0(\lambda_n r)\; dr$$

$$= -\frac{1}{\lambda_n} \left[r^2 j_1(\lambda_n r)\, j_0(\lambda_n r) \right]_0^1 + \frac{1}{\lambda_n} \int_0^1 j_0(\lambda_n r) \frac{d}{dr} \left[r^2 j_1(\lambda_n r) \right]\; dr$$

$$= \frac{1}{\lambda_n^3} \int_0^1 \frac{\sin(\lambda_n r)}{r} \frac{d}{dr} \left[\frac{\sin(\lambda_n r)}{\lambda_n} - r\cos(\lambda_n r) \right]\; dr$$

$$= \frac{1}{\lambda_n^3} \int_0^1 \frac{\sin(\lambda_n r)}{r} \left[\cos(\lambda_n r) - \cos(\lambda_n r) + r\lambda_n \sin(\lambda_n r) \right]\; dr$$

$$= \frac{1}{\lambda_n^2} \int_0^1 \sin^2(\lambda_n r)\; dr = \frac{1}{2\lambda_n^2} \int_0^1 1 - \cos(2\lambda_n r)\; dr$$

$$= \frac{2\lambda_n - \sin(2\lambda_n)}{4\lambda_n^3} = \frac{\lambda_n - \sin(\lambda_n)\cos(\lambda_n)}{2\lambda_n^3} = \frac{\sin^2(\lambda_n)}{2\lambda_n^2} = \tfrac{1}{2} j_0^2(\lambda_n).$$