<u>Solutions for Math 436 2015 Midterm</u>

Question 1: (a) The $1st$-order pde is given by

$$u_t + c\,u_x = -u^2.$$

To obtain the general solution we need the characteristic variable. The characteristics are the level curves associated with

$$\frac{dx}{dt} = c \Longrightarrow \xi = x - ct,$$

and we take the additional independent variable $\eta = x$ (one could take any other independent variable). The derivatives map according to

$$\partial_t = -c\partial_\xi \text{ and } \partial_x = \partial_\xi + \partial_\eta.$$

Thus, the pde maps to

$$-cu_\xi + c\left(u_\xi + u_\eta\right) = -u^2 \Longrightarrow -cu_\eta = u^2$$

$$\Longrightarrow \frac{c}{u} = \eta + \phi\left(\xi\right) \Longrightarrow u = \frac{c}{\eta + \phi\left(\xi\right)},$$

where $\phi\left(\xi\right)$ is an arbitrary function of its argument. Thus, the general solution $u\left(x,t\right)$ is given by

$$u\left(x,t\right) = \frac{c}{x + \phi\left(x - ct\right)}.$$

(b) If we impose the initial condition $u\left(x,0\right) = x$, then

$$\frac{c}{x + \phi\left(x\right)} = x \Longrightarrow \phi\left(x\right) = \frac{c}{x} - x.$$

Hence

$$u\left(x,t\right) = \frac{c}{x + \frac{c}{x-ct} - \left(x - ct\right)} = \frac{x - ct}{1 + t\left(x - ct\right)}.$$

Question 2: The pde is given by

$$tv_x + xv_t = cv \text{ with } v\left(x,x\right) = f\left(x\right).$$

The first thing we must verify is that the line $x = t$ is a characteristic. The characteristics are the level curves associated with

$$\frac{dx}{dt} = \frac{t}{x} \Longrightarrow \xi = x^2 - t^2,$$

where ξ is any constant. We see that $x = t$ is a characteristic corresponding to $\xi = 0$. It therefore follows that we require

$$\frac{df}{dx} = v_x\left(x,x\right) + v_t\left(x,x\right)\cdot 1 = \frac{xv_x\left(x,x\right) + xv_t\left(x,x\right)}{x} = \frac{cf}{x}$$

$$\Longrightarrow \ln f = c \ln x + \ln \alpha \Longrightarrow f(x) = \alpha x^c,$$

for any constant α.

Question 3: The shallow water equations are given by

$$h_t + u h_x + h u_x = 0,$$

$$u_t + u\, u_x + g h_x = 0.$$

These can be written in the matrix form

$$\begin{bmatrix} u & h \\ g & u \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_x + \begin{bmatrix} h \\ u \end{bmatrix}_t = \mathbf{0}.$$

To classify, we examine the eigenvalue problem

$$\begin{bmatrix} u & h \\ g & u \end{bmatrix}^\top \mathbf{v} = \lambda \mathbf{v} \Longleftrightarrow \begin{bmatrix} u-\lambda & g \\ h & u-\lambda \end{bmatrix} \mathbf{v} = \mathbf{0} \Longrightarrow \lambda = u \pm \sqrt{gh}.$$

Since both λ are real and distinct we have a hyperbolic system. To reduce to characteristic normal form we need the eigenvectors. For $\lambda = u + \sqrt{gh}$, we have

$$\begin{bmatrix} -\sqrt{gh} & g \\ h & -\sqrt{gh} \end{bmatrix} \mathbf{v} = \mathbf{0} \Longrightarrow \mathbf{v} = \begin{bmatrix} \sqrt{g} \\ \sqrt{h} \end{bmatrix},$$

and for $\lambda = u - \sqrt{gh}$, we have

$$\begin{bmatrix} \sqrt{gh} & g \\ h & \sqrt{gh} \end{bmatrix} \mathbf{v} = \mathbf{0} \Longrightarrow \mathbf{v} = \begin{bmatrix} \sqrt{g} \\ -\sqrt{h} \end{bmatrix}.$$

The characteristic normal form equation associated with $\lambda = u + \sqrt{gh}$ will be determined by

$$\begin{bmatrix} \sqrt{g} & \sqrt{h} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} \sqrt{g} & \sqrt{h} \end{bmatrix} \begin{bmatrix} u & h \\ g & u \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_x = 0$$

$$\Longleftrightarrow \begin{bmatrix} \sqrt{g} & \sqrt{h} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} \sqrt{g}u + g\sqrt{h} & h\sqrt{g} + u\sqrt{h} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_x = 0$$

$$\sqrt{g}\left[h_t + \left(u + \sqrt{gh}\right) h_x\right] + \sqrt{h}\left[u_t + \left(u + \sqrt{gh}\right) u_x\right] = 0.$$

The characteristic normal form equation associated with $\lambda = u - \sqrt{gh}$ will be determined by

$$\begin{bmatrix} \sqrt{g} & -\sqrt{h} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} \sqrt{g} & -\sqrt{h} \end{bmatrix} \begin{bmatrix} u & h \\ g & u \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_x = 0$$

$$\Longleftrightarrow \begin{bmatrix} \sqrt{g} & -\sqrt{h} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} \sqrt{g}u - g\sqrt{h} & h\sqrt{g} - u\sqrt{h} \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_x = 0$$

$$\sqrt{g}\left[h_t + \left(u - \sqrt{gh}\right)h_x\right] - \sqrt{h}\left[u_t + \left(u - \sqrt{gh}\right)u_x\right] = 0.$$

Question 4: To classify the linear pde

$$u_{xx} + 4u_{xy} + 3u_{yy} + u = 0,$$

we examine the $\omega^{\pm}$ roots given by

$$\omega^{\pm} = \frac{-4 \pm \sqrt{16-12}}{2} = \frac{-4 \pm 2}{2} \Longrightarrow \omega^{-} = -3 \text{ and } \omega^{+} = -1.$$

Since $16 - 12 = 4 > 0$, the pde is hyperbolic $\forall\ (x, y) \in \mathbb{R}^2$. The characteristic variables are determined by

$$\left(\frac{dy}{dx}\right)_{\xi} = -\omega^{+} = 1 \Longrightarrow \xi = y - x,$$

$$\left(\frac{dy}{dx}\right)_{\eta} = -\omega^{-} = 3 \Longrightarrow \eta = y - 3x.$$

The derivatives will transform according to

$$u_x = u_\xi \xi_x + u_\eta \eta_x = -\left(\partial_\xi + 3\partial_\eta\right)u,$$

$$u_{xx} = \left(\partial_\xi + 3\partial_\eta\right)^2 u = \left(\partial_{\xi\xi} + 6\partial_{\xi\eta} + 9\partial_{\eta\eta}\right)u,$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = \left(\partial_\xi + \partial_\eta\right)u,$$

$$u_{yy} = \left(\partial_\xi + \partial_\eta\right)^2 u = \left(\partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}\right)u,$$

$$u_{xy} = -\left(\partial_\xi + 3\partial_\eta\right)\left(\partial_\xi + \partial_\eta\right)u = -\left(\partial_{\xi\xi} + 4\partial_{\xi\eta} + 3\partial_{\eta\eta}\right)u.$$

Thus the pde maps to

$$\left(\partial_{\xi\xi} + 6\partial_{\xi\eta} + 9\partial_{\eta\eta}\right)u - 4\left(\partial_{\xi\xi} + 4\partial_{\xi\eta} + 3\partial_{\eta\eta}\right)u + 3\left(\partial_{\xi\xi} + 2\partial_{\xi\eta} + \partial_{\eta\eta}\right)u + u = 0$$

which reduces to the (H1) canonical form

$$u_{\xi\eta} - \tfrac{1}{4}u = 0.$$