Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

15 December 2015

Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Consider the linear partial differential operator L defined by

$$Lu = -\nabla \cdot (p \nabla u) + qu, \text{ for } x \in G \subset \mathbb{R}^n,$$

where G is an open, simply-connected bounded region with smooth boundary ∂G, $p = p(x) > 0$ and $q = q(x) \geq 0$, with the boundary condition

$$\alpha(x) u + \beta(x) \frac{\partial u}{\partial n} = 0, \text{ where } \alpha, \beta \geq 0,\ \alpha + \beta > 0,\ x \in \partial G,$$

and where the inner product is given by

$$(u, w) \equiv \int_G \rho\, u\, w\, dx, \text{ where } \rho = \rho(x) > 0.$$

   (a) Define a self-adjoint operator, and show that $\frac{1}{\rho}L$ is self-adjoint.

   (b) Define a positive operator, and show that $\frac{1}{\rho}L$ is positive.

   (c) Show that the eigenvalues of $\frac{1}{\rho}L$ are non-negative.

2. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is an orthonormal sequence of square-integrable functions defined on $x \in G \subset \mathbb{R}^n$ with the inner product

$$(u, w) \equiv \int_G \rho\, u\, w\, dx, \text{ with } \rho = \rho(x) > 0.$$

   (a) If $\varphi(x)$ is a square-integrable function for $x \in G$, define the *Fourier Series* for $\varphi(x)$ with respect to $\{\varphi_k(x)\}_{k=1}^{\infty}$.

   (b) Beginning with the n^{th} partial sum associated with the Fourier Series for $\varphi(x)$, denoted by $\psi_n(x)$, show that *Bessel's Inequality* holds, i.e.,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2 .$$

   (c) Define what it means for a sequence of square-integrable functions $\{\psi_n(x)\}_{n=1}^{\infty}$ to *converge* to a function $\varphi(x)$ *in the mean*.

   (d) Show that *convergence in the mean* is equivalent to *Parseval's Identity*.

3. Let $\mathcal{L}$ be a *positive, self-adjoint, real-valued partial differential operator* defined for smooth square-integrable functions $f(x)$ where $x \in G \subset \mathbb{R}^n$ and satisfying the boundary conditions

$$\alpha f + \beta \frac{\partial f}{\partial n} = 0, \text{ with } \alpha, \beta \geq 0 \text{ where } \alpha + \beta > 0, \text{ for } x \in \partial G.$$

Show that the solution, assuming it exists, to

$$u_{tt} + \mathcal{L}u = F(x,t), \; x \in G, \; t > 0,$$

$$u(x,0) = f(x), \; u_t(x,0) = g(x) \text{ for } x \in G,$$

$$\text{and } \alpha u + \beta \frac{\partial u}{\partial n} = B(x,t) \text{ for } x \in \partial G, \; t > 0,$$

is unique. HINT: Assume a unit density function in the inner product.

4. Determine the set of orthonormalized eigenfunctions and corresponding eigenvalues for the positive self-adjoint operator

$$L = -\frac{d^2}{dx^2}, \quad 0 < x < 1,$$

assuming homogenous Dirichlet boundary conditions at $x = 0$ and 1, respectively, with respect to the inner product with unit density function.