Solutions for Math 436 2015 Final

Question 1a: Let $f(x)$ and $g(x)$ be smooth square-integrable functions that satisfy the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be self-adjoint if

$$(f,\,\mathcal{L}g) = (g,\,\mathcal{L}f)\,.$$

To show $\frac{1}{\rho}L$ is self-adjoint, we will show that

$$\left(f,\,\frac{1}{\rho}Lg\right) - \left(g,\,\frac{1}{\rho}Lf\right) = 0.$$

We have

$$\left(f,\,\frac{1}{\rho}Lg\right) - \left(g,\,\frac{1}{\rho}Lf\right) = \int\limits_G f\,Lg - g\,Lf\;dx$$

$$= \int\limits_G f\,\left[-\nabla\cdot(p\nabla g) + qg\right] - g\,\left[-\nabla\cdot(p\nabla f) + qf\right]\;dx$$

$$= \int\limits_G g\,\nabla\cdot(p\nabla f) - f\,\nabla\cdot(p\nabla g)\;dx$$

$$= \int\limits_{\partial G} g\,\mathbf{n}\cdot(p\nabla f) - f\,\mathbf{n}\cdot(p\nabla g)\;dx + \int\limits_G p\,\left[\nabla f\cdot\nabla g - \nabla g\cdot\nabla f\right]\;dx$$

$$= \int\limits_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\,dx.$$

If $\beta = 0$, then $f = g = 0$ for $x \in \partial G$ so that

$$\int\limits_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\,dx = 0.$$

If $\beta \neq 0$, then

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ and } \frac{\partial g}{\partial n} = -\frac{\alpha g}{\beta} \text{ for } x \in \partial G$$

$$\Longrightarrow \int\limits_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\,dx = \int\limits_{\partial G} p\left(\frac{\alpha g f}{\beta} - \frac{\alpha g f}{\beta}\right)\,dx = 0.$$

Thus we have shown that

$$\left(f,\,\frac{1}{\rho}Lg\right) = \left(g,\,\frac{1}{\rho}Lf\right).$$

Question 1b: Let $f(x)$ be a smooth square-integrable function that satisfies the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be positive if

$$(f,\,\mathcal{L}f) \geq 0.$$

We proceed directly:

$$\left(f,\,\frac{1}{\rho}Lf\right)=\int\limits_G f\,Lf\;dx=\int\limits_G f\;\left[-\nabla\cdot(p\nabla f)+qf\right]\;dx$$

$$=-\int\limits_{\partial G} pf\frac{\partial f}{\partial n}\,dx+\int\limits_G p\,\nabla f\cdot\nabla f+q\,f^2\;dx.$$

If $\beta=0$, then $f=0$ for $x\in\partial G$ so that

$$\left(f,\,\frac{1}{\rho}Lf\right)=\int\limits_G p\,\nabla f\cdot\nabla f+q\,f^2\;dx\geq 0,$$

since $p>0$ and $q\geq 0$. If $\beta\neq 0$, then

$$\frac{\partial f}{\partial n}=-\frac{\alpha f}{\beta}\text{ for }x\in\partial G$$

$$\Longrightarrow\left(f,\,\frac{1}{\rho}Lf\right)=\int\limits_{\partial G}\frac{\alpha p f^2}{\beta}\,dx+\int\limits_G p\,\nabla f\cdot\nabla f+q\,f^2\;dx\geq 0,$$

since $p>0$, $q\geq 0$, $\alpha\geq 0$ and $\beta>0$.

Question 1c: The eigenvalue problem is given by

$$\frac{1}{\rho}Lu=\lambda u,\;x\in G,$$

with the boundary condition

$$\alpha\left(x\right)u+\beta\left(x\right)\frac{\partial u}{\partial n}=0\text{ for }x\in\partial G.$$

Since $\frac{1}{\rho}L$ is a positive operator

$$0\leq\left(u,\,\frac{1}{\rho}Lu\right)=\lambda\left(u,\,u\right)=\lambda\left\|u\right\|^2\Longrightarrow\lambda\geq 0.$$

Question 2a: The Fourier Series is defined as

$$\varphi\left(x\right)=\sum_{k=1}^{\infty}\left(\varphi,\varphi_k\right)\;\varphi_k\left(x\right).$$

Question 2b: The n^{th} partial sum is given by

$$\psi_n\left(x\right)=\sum_{k=1}^{n}\left(\varphi,\varphi_k\right)\;\varphi_k\left(x\right).$$

To show Bessel's Inequality, we begin with

$$0 \leq \|\varphi(x) - \psi_n(x)\|^2 = (\varphi - \psi_n, \varphi - \psi_n) = (\varphi, \varphi) - 2(\varphi, \psi_n) + (\psi_n, \psi_n)$$

$$= (\varphi, \varphi) - 2\left(\varphi, \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right) + \left(\sum_{m=1}^{n} (\varphi, \varphi_m)\ \varphi_m(x), \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi,\ \varphi_k) + \sum_{m=1}^{n}\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_m)(\ \varphi_m,\ \varphi_k)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)^2 + \sum_{k=1}^{n} (\varphi, \varphi_k)^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2$$

$$\Longrightarrow \sum_{k=1}^{n} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit $n \to \infty$, it follows

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).$$

Question 2c: Mean square convergence is defined as

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

Question 2d: We must show that

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi).$$

From Question 2b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Thus, provided the limit exists,

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \lim_{n\to\infty} \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Hence

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0 \Longrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi),$$

and

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) \Longrightarrow \lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0.$$

Question 3: Assume two solutions exist to the problem denoted by $u_1(x,t)$ and $u_2(x,t)$, respectively, i.e.,

$$\partial_{tt} u_1 + \mathcal{L} u_1 = F(x,t), \; x \in G, \; t > 0,$$

$$u_1(x,0) = f(x), \;\; \partial_t u_1(x,0) = g(x) \;\; \text{for } x \in G,$$

$$\text{and } \alpha u_1 + \beta \frac{\partial u_1}{\partial n} = B(x,t) \;\; \text{for } x \in \partial G, \; t > 0,$$

and

$$\partial_{tt} u_2 + \mathcal{L} u_2 = F(x,t), \; x \in G, \; t > 0,$$

$$u_2(x,0) = f(x), \;\; \partial_t u_2(x,0) = g(x) \;\; \text{for } x \in G,$$

$$\text{and } \alpha u_2 + \beta \frac{\partial u_2}{\partial n} = B(x,t) \;\; \text{for } x \in \partial G, \; t > 0.$$

Define the difference $w = u_1 - u_2$, it follows that w satisfies

$$w_{tt} + \mathcal{L} w = 0, \; x \in G, \; t > 0,$$

$$w(x,0) = 0, \;\; w_t(x,0) = 0 \text{ for } x \in G,$$

$$\text{and } \alpha w + \beta \frac{\partial w}{\partial n} = 0 \text{ for } x \in \partial G, \; t > 0.$$

We form the energy equation

$$w_t w_{tt} + w_t \mathcal{L} w = 0 \Longrightarrow \partial_t \|w_t\|^2 + 2(w_t, \mathcal{L} w) = 0,$$

and since $\mathcal{L}$ is a *self-adjoint operator* it follows that $(w_t, \mathcal{L} w) = (w, \mathcal{L} w_t)$, so that the energy equation can be written in the form

$$\partial_t \|w_t\|^2 + (w_t, \mathcal{L} w) + (w, \mathcal{L} w_t) = 0 \Longrightarrow \frac{\partial}{\partial t}\left[\|w_t\|^2 + (w, \mathcal{L} w)\right] = 0$$

$$\Longrightarrow \|w_t\|^2 + (w, \mathcal{L} w) = \left[\|w_t\|^2 + (w, \mathcal{L} w)\right]_{t=0} = 0$$

$$\Longrightarrow \|w_t\|^2 = -(w, \mathcal{L} w) \le 0,$$

since $\mathcal{L}$ is a *positive operator*, i.e., $(w, \mathcal{L} w) \ge 0$. Hence

$$\|w_t\| = 0 \Longrightarrow w_t = 0 \Longrightarrow w(x,t) = w(x,0) = 0.$$

Question 4: The eigenvalue problem is given by

$$-u_{xx} = \lambda^2 u \text{ for } 0 < x < 1,$$

subject to

$$u(0) = u(1) = 0.$$

The general solution is given by

$$u = A\cos(\lambda x) + B\sin(\lambda x).$$

Application of the boundary conditions leads to

$$u(0) = 0 = A,$$

$$u(1) = 0 = B\sin(\lambda) \Longrightarrow \lambda = n\pi \text{ for } n = 1, 2, 3, ...$$

Thus, the orthonormalized eigenfunctions are given by

$$u_n(x) = \frac{\sin(n\pi x)}{\left[\int_0^1 \sin^2(n\pi x)\, dx\right]^{\frac{1}{2}}},$$

where

$$\int_0^1 \sin^2(n\pi x)\, dx = \int_0^1 \frac{1-\cos(2n\pi x)}{2}\, dx = \frac{1}{2}.$$

Hence

$$u_n(x) = \sqrt{2}\sin(n\pi x).$$