Solutions for Math 436 2014 Midterm

Question 1: The initial data curve is along the x-axis, which we may parameterize as $x = \tau$ and $t = 0$. The characteristic equations are therefore

$$\frac{dt}{ds} = 1 \text{ subject to } \left. t\right|_{s=0} = 0,$$

$$\frac{dx}{ds} = u \text{ subject to } \left. x\right|_{s=0} = \tau,$$

$$\frac{du}{ds} = 0 \text{ subject to } \left. u\right|_{s=0} = f(\tau),$$

$$\frac{d\rho}{ds} = -\rho\, u_x \text{ subject to } \left. \rho\right|_{s=0} = g(\tau).$$

The equation for u can be integrated to yield

$$u = f(\tau),$$

which allows the equation for x to be integrated to yield

$$x = sf(\tau) + \tau,$$

and the solution for t is given by

$$t = s.$$

In order to substitute into the characteristic equation for ρ we need to compute u_x (as a function of s and τ) which is given by

$$u_x = f'(\tau)\,\tau_x = \frac{f'(\tau)}{1 + s\, f'(\tau)},$$

where prime means differentiation with respect to the argument. Thus,

$$\frac{d\rho}{ds} = -\frac{\rho\, f'(\tau)}{1 + sf'(\tau)} \Longrightarrow \int \frac{d\rho}{\rho} = -\int \frac{f'(\tau)}{1 + sf'(\tau)}\, ds$$

$$\Longrightarrow \ln\rho - \ln g(\tau) = -\ln\left(1 + s\, f'(\tau)\right) \Longrightarrow \rho = \frac{g(\tau)}{1 + s\, f'(\tau)}.$$

Finally, eliminating s from the solution for x allows us to write $\tau = x - ut$, so that we can write the implicit solution

$$u = f(x - ut),$$

$$\rho = \frac{g(x - ut)}{1 + t\, f'(x - ut)}.$$

Question 2: The pde and initial condition, given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)\,,$$

$$u(x,h(x)) = f(x)\,,$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x,h)}{a(x,h)}.$$

It follows that

$$\frac{df}{dx} = u_x(x,h(x)) + u_y(x,h(x))\,\frac{dh}{dx}$$

$$= u_x(x,h(x)) + \frac{b(x,h)}{a(x,h)} u_y(x,h(x)) = \frac{a(x,h)\,u_x(x,h(x)) + b(x,h)\,u_y(x,h(x))}{a(x,h)}$$

$$= \frac{c(x,h)\,u(x,h(x)) + d(x,h)}{a(x,h)} = \frac{c(x,h)\,f + d(x,h)}{a(x,h)}.$$

Question 3: The pde and initial conditions are given by

$$u_{tt} - u_{xx} = h(x,t)\,, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x)\,, \quad -\infty < x < \infty,$$

where $h(x,t)$, $f(x)$ and $g(x)$ are smooth and spatially square-integrable functions. To show uniqueness, we assume that there are two solutions, given by $u_1(x,t)$ and $u_2(x,t)$, i.e.,

$$(\partial_{tt} - \partial_{xx})\,u_1 = h, \quad -\infty < x < \infty,\ t > 0,$$

$$u_1(x,0) = f \text{ and } \partial_t u_1(x,0) = g, \quad -\infty < x < \infty,$$

and

$$(\partial_{tt} - \partial_{xx})\,u_2 = h, \quad -\infty < x < \infty,\ t > 0,$$

$$u_2(x,0) = f \text{ and } \partial_t u_2(x,0) = g, \quad -\infty < x < \infty.$$

Let $\Phi(x,t) = u_1(x,t) - u_2(x,t)$. We will show that $\Phi(x,t) = 0$ for all $t \geq 0$. Hence $u_1(x,t) = u_2(x,t)$ and we have established uniqueness. It follows that

$$(\partial_{tt} - \partial_{xx})\,\Phi = 0, \quad -\infty < x < \infty,\ t > 0, \tag{1}$$

$$\Phi(x,0) = 0 \text{ and } \Phi_t(x,0) = 0, \quad -\infty < x < \infty. \tag{2}$$

The energy equation associated is obtained by multiplying (1) by Φ_t and re-writing the resulting equation as a space-time divergence, i.e.,

$$\Phi_t(\partial_{tt} - \partial_{xx})\,\Phi = \frac{1}{2}\partial_t\left[(\Phi_t)^2 + (\Phi_x)^2\right] - \partial_x(\Phi_t \Phi_x) = 0.$$

It therefore follows that

$$\partial_t \int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \, dx = 2\Phi_t\Phi_x|_{-\infty}^{\infty} = 0, \tag{3}$$

since $\Phi_t\Phi_x \to 0$ as $|x| \to \infty$ since $\Phi_{x,t}$ are smooth square-integrable functions. Thus, it follows from (3) that

$$\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \, dx = \left[\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2 \, dx\right]_{t=0} = 0, \tag{4}$$

since $\Phi(x,0) = 0$ ($\Longrightarrow \Phi_x(x,0) = 0$) and $\Phi_t(x,0) = 0$. Further, it then follows from (4) that

$$\Phi_t(x,t) = \Phi_x(x,t) = 0 \text{ for all } t \geq 0 \Longrightarrow \Phi(x,t) = 0 \text{ for all } t \geq 0.$$

Question 4: The pde can be written in the matrix form

$$\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} A \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u = 0 \text{ where } A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^3$ of A are given by

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$$

$$\Longrightarrow \lambda_1 = 1 > 0, \, \lambda_2 = 2 > 0 \text{ and } \lambda_3 = 0.$$

Hence, we conclude that the pde is *parabolic*. To map the pde into canonical form we first need the orthonormal eigenbasis vectors, denoted by $\mathbf{r}_{1,2,3}$, and determined from

$$(A - \lambda_i I)\,\mathbf{r}_i = \mathbf{0}.$$

We therefore obtain

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \mathbf{r}_1 = \mathbf{0} \Longrightarrow \mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \mathbf{r}_2 = \mathbf{0} \Longrightarrow \mathbf{r}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{r}_3 = \mathbf{0} \Longrightarrow \mathbf{r}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We now introduce the new coordinates (α, β, μ) given by

$$\alpha = \frac{\mathbf{r}_1^\top \cdot \mathbf{x}}{\sqrt{|\lambda_1|}} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y,$$

$$\beta = \frac{\mathbf{r}_2^\top \cdot \mathbf{x}}{\sqrt{|\lambda_2|}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{x-z}{2},$$

$$\mu = \mathbf{r}_3^\top \cdot \mathbf{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{x+z}{\sqrt{2}}.$$

Consequently, derivatives map according to

$$u_x = \frac{1}{2}\left(u_\beta + u_\mu\right) \Longrightarrow u_{xx} = \frac{1}{4}\left(u_{\beta\beta} + 2u_{\beta\mu} + u_{\mu\mu}\right),$$

$$u_z = \frac{1}{2}\left(-u_\beta + u_\mu\right) \Longrightarrow u_{zz} = \frac{1}{4}\left(u_{\beta\beta} - 2u_{\beta\mu} + u_{\mu\mu}\right) \text{ and } u_{xz} = \frac{1}{4}\left(-u_{\beta\beta} + u_{\mu\mu}\right),$$

$$u_{yy} = u_{\alpha\alpha}.$$

Finally, substitution into the pde yields

$$u_{\alpha\alpha} + u_{\beta\beta} = 0.$$