Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Consider the 2×2 system of pdes is given by

$$\mathbf{u}_y + A\mathbf{u}_x = \mathbf{0} \text{ where } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

   (a) Classify the system of pdes.

   (b) Reduce the system of pdes to canonical form.

2. Consider the pde

$$u_{tt} + 2u_{xt} - a\, u_{xx} + u = 0,\ x, a \in \mathbb{R},\ t > 0.$$

   (a) Determine the stability index Ω as a function of the parameter a.

   (b) Determine the stability of the pde as a function of the parameter a.

   (c) Determine the values of the parameter a for which the Cauchy problem is ill-posed.

3. Consider the linear partial differential operator L defined by

$$Lu = -\nabla \cdot (p\nabla u) + qu, \text{ for } x \in G \subset \mathbb{R}^n,$$

   where G is an open, simply-connected bounded region with smooth boundary ∂G, $p = p(x) > 0$ and $q = q(x) \geq 0$, with the boundary condition

$$\alpha(x)\, u + \beta(x) \frac{\partial u}{\partial n} = 0, \text{ where } \alpha, \beta \geq 0,\ \ \alpha + \beta > 0,\ x \in \partial G,$$

   and where the inner product is given by

$$(u, w) \equiv \int_G \rho\, u\, w\ dx, \text{ where } \rho = \rho(x) > 0.$$

   (a) Define a self-adjoint operator, and show that $\frac{1}{\rho}L$ is self-adjoint.

   (b) Define a positive operator, and show that $\frac{1}{\rho}L$ is positive.

   (c) Show that the eigenvalues of $\frac{1}{\rho}L$ are non-negative.

4. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is an orthonormal sequence of square-integrable functions defined on $x \in G \subset \mathbb{R}^n$ with the inner product

$$(u, w) \equiv \int_G \rho\, u\, w\ dx, \text{ with } \rho = \rho(x) > 0.$$

(a) If $\varphi(x)$ is a square-integrable function for $x \in G$, define the *Fourier Series* for $\varphi(x)$ with respect to $\{\varphi_k(x)\}_{k=1}^{\infty}$.

(b) Beginning with the n^{th} partial sum associated with the Fourier Series for $\varphi(x)$, denoted by $\psi_n(x)$, show that *Bessel's Inequality* holds, i.e.,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2 .$$

(c) Define what it means for a sequence of square-integrable functions $\{\psi_n(x)\}_{n=1}^{\infty}$ to *converge* to a function $\varphi(x)$ *in the mean.*

(d) Show that *convergence in the mean* is equivalent to *Parseval's Identity.*