Solutions for Math 436 2014 Final

Question 1a: If $y = h(x)$ are the characteristics, then $h(x)$ must satisfy

$$|I - h'(x)\,A| = \begin{vmatrix} 1-h' & -2h' \\ -2h' & 1-h' \end{vmatrix} = (1-h')^2 - 4(h')^2 = 0$$

$$\Longrightarrow 1 - h' = \pm 2h' \Longrightarrow h' = \frac{1}{3} \text{ or } -1.$$

Since all of the h' are real and distinct, *the system is strictly or totally hyperbolic.*
Question 1b: To reduce to canonical form we first need to compute the right eigenvectors associated with h' computed in part a. The *right eigenvectors* of A associated with the eigenvalues $1/h'$, i.e., -1 and 3, and denoted by $\mathbf{r}_{-1}$ and $\mathbf{r}_3$, respectively, are determined by

$$(I + A)\,\mathbf{r}_{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{r}_{-1} = \mathbf{0} \Longrightarrow \mathbf{r}_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\left(I - \tfrac{1}{3}A\right)\mathbf{r}_3 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \mathbf{r}_3 = \mathbf{0} \Longrightarrow \mathbf{r}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that $\mathbf{r}_{-1} \cdot \mathbf{r}_3 = 0$. Thus, introducing the transformation

$$\mathbf{u} \equiv R\mathbf{v} = R\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \text{ where } R \equiv \begin{bmatrix} \mathbf{r}_{-1} & \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\Longrightarrow R^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

into the pde, leads to

$$\mathbf{v}_y + D\mathbf{v}_x = \mathbf{0}, \text{ where } D \equiv R^{-1}AR = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix},$$

or, in component form,

$$\partial_y v_1 - \partial_x v_1 = 0 \text{ and } \partial_y v_2 + 3\partial_x v_2 = 0.$$

Question 2a: To determine the stability index Ω, we substitute

$$u = A\exp(ikx + \lambda t) + c.c.,$$

into the the pde

$$u_{tt} + 2u_{xt} - a\,u_{xx} + u = 0,$$

to yield

$$\left(\lambda^2 + 2ik\lambda + ak^2 + 1\right) A\exp(ikx + \lambda t) + c.c. = 0.$$

For non-trivial $(A \neq 0)$ solutions

$$\lambda^2 + 2ik\lambda + ak^2 + 1 = 0$$

$$\Longrightarrow \lambda = -ik \pm \sqrt{-1-(1+a)\,k^2} = -ik \pm i\sqrt{1+(1+a)\,k^2}.$$

It therefore follows that

$$\Omega = lub_k \operatorname{Re}[\lambda(k)] = \begin{cases} 0 \text{ if } a \geq -1 \\ +\infty \text{ if } a < -1. \end{cases}$$

Question 2b: The pde is *neutrally stable* if $a \geq -1$ and is *unstable* if $a < -1$.
Question 2c: The Cauchy problem is ill-posed if $a < -1$.
Question 3a: Let $f(x)$ and $g(x)$ be smooth square-integrable functions that satisfy the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be self-adjoint if

$$(f,\, \mathcal{L}g) = (g,\, \mathcal{L}f)\,.$$

To show $\frac{1}{\rho}L$ is self-adjoint, we will show that

$$\left(f,\, \frac{1}{\rho}Lg\right) - \left(g,\, \frac{1}{\rho}Lf\right) = 0.$$

We have

$$\left(f,\, \frac{1}{\rho}Lg\right) - \left(g,\, \frac{1}{\rho}Lf\right) = \int\limits_G f\,Lg - g\,Lf\ dx$$

$$= \int\limits_G f\ [-\nabla\cdot(p\nabla g) + qg] - g\ [-\nabla\cdot(p\nabla f) + qf]\ dx$$

$$= \int\limits_G g\,\nabla\cdot(p\nabla f) - f\,\nabla\cdot(p\nabla g)\ dx$$

$$= \int\limits_{\partial G} g\,\mathbf{n}\cdot(p\nabla f) - f\,\mathbf{n}\cdot(p\nabla g)\ dx + \int\limits_G p\ [\nabla f\cdot\nabla g - \nabla g\cdot\nabla f]\ dx$$

$$= \int\limits_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\ dx.$$

If $\beta = 0$, then $f = g = 0$ for $x \in \partial G$ so that

$$\int\limits_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\ dx = 0.$$

If $\beta \neq 0$, then

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ and } \frac{\partial g}{\partial n} = -\frac{\alpha g}{\beta} \text{ for } x \in \partial G$$

$$\Longrightarrow \int\limits_{\partial G} p\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)\ dx = \int\limits_{\partial G} p\left(\frac{\alpha g f}{\beta} - \frac{\alpha g f}{\beta}\right)\ dx = 0.$$

Thus we have shown that

$$\left(f,\ \frac{1}{\rho}Lg\right) = \left(g,\ \frac{1}{\rho}Lf\right).$$

Question 3b: Let $f(x)$ be a smooth square-integrable function that satisfies the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be positive if

$$(f,\ \mathcal{L}f) \geq 0.$$

To show $\frac{1}{\rho}L$ is positive, we proceed directly:

$$\left(f,\ \frac{1}{\rho}Lf\right) = \int\limits_G f\,Lf\ dx = \int\limits_G f\ \left[-\nabla\cdot(p\nabla f) + qf\right]\ dx$$

$$= -\int\limits_{\partial G} pf\frac{\partial f}{\partial n}\,dx + \int\limits_G p\,\nabla f\cdot\nabla f + q\,f^2\ dx.$$

If $\beta = 0$, then $f = 0$ for $x \in \partial G$ so that

$$\left(f,\ \frac{1}{\rho}Lf\right) = \int\limits_G p\,\nabla f\cdot\nabla f + q\,f^2\ dx \geq 0,$$

since $p > 0$ and $q \geq 0$. If $\beta \neq 0$, then

$$\frac{\partial f}{\partial n} = -\frac{\alpha f}{\beta} \text{ for } x \in \partial G$$

$$\Longrightarrow \left(f,\ \frac{1}{\rho}Lf\right) = \int\limits_{\partial G} \frac{\alpha p f^2}{\beta}\,dx + \int\limits_G p\,\nabla f\cdot\nabla f + q\,f^2\ dx \geq 0,$$

since $p > 0$, $q \geq 0$, $\alpha \geq 0$ and $\beta > 0$.

Question 3c: The eigenvalue problem is given by

$$\frac{1}{\rho}Lu = \lambda u,\ x \in G,$$

with the boundary condition

$$\alpha(x)\,u + \beta(x)\,\frac{\partial u}{\partial n} = 0 \text{ for } x \in \partial G.$$

Since $\frac{1}{\rho}L$ is a positive operator

$$0 \leq \left(u,\ \frac{1}{\rho}Lu\right) = \lambda\,(u,\ u) = \lambda\,\|u\|^2 \Longrightarrow \lambda \geq 0.$$

Question 4a: The Fourier Series is defined as

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k)\ \varphi_k(x).$$

Question 4b: The n^{th} partial sum is given by

$$\psi_n(x) = \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x).$$

To show Bessel's Inequality, we begin with

$$0 \le \|\varphi(x) - \psi_n(x)\|^2 = (\varphi - \psi_n, \varphi - \psi_n) = (\varphi, \varphi) - 2(\varphi, \psi_n) + (\psi_n, \psi_n)$$

$$= (\varphi, \varphi) - 2\left(\varphi, \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right) + \left(\sum_{m=1}^{n} (\varphi, \varphi_m)\ \varphi_m(x), \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_k) + \sum_{m=1}^{n}\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_m)(\varphi_m, \varphi_k)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)^2 + \sum_{k=1}^{n} (\varphi, \varphi_k)^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2$$

$$\Longrightarrow \sum_{k=1}^{n} (\varphi, \varphi_k)^2 \le (\varphi, \varphi).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit $n \to \infty$, it follows

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \le (\varphi, \varphi).$$

Question 4c: Mean square convergence is defined as

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

Question 4d: We must show that

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi).$$

From Question 4b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Thus, provided the limit exists,

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \lim_{n\to\infty} \sum_{k=1}^{n} (\varphi, \varphi_k)^2 .$$

Hence

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0 \Longrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) ,$$

and

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) \Longrightarrow \lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0.$$