Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

2 November 2010

Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Consider the following quasi-linear $1st$-order pde:

$$u_t + u\,u_x = 0,\ \ x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \begin{cases} 1 \text{ for } x \leq 0, \\ 1 - x/a \text{ for } 0 < x \leq a, \\ 0 \text{ for } x > a. \end{cases}$$

   (a) Using the Method of Characteristics, find the solution $u(x,t)$.

   (b) Determine the space-time coordinates (x_s, t_s) of first shock formation.

   (c) Once the shock has formed, use the entropy condition to find the position of the shock for $t \geq t_s$.

   (d) Finally, determine the solution for $t \geq t_s$.

2. Suppose the initial data for the linear $1st$-order pde

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)\,,$$

   is given by

$$u(x, h(x)) = f(x)\,,$$

   where $y = h(x)$ is a characteristic. Show that

$$\frac{df}{dx} = \frac{c(x,h)\,f + d(x,h)}{a(x,h)}.$$

3. Consider the following hyperbolic Cauchy problem:

$$u_{tt} - u_{xx} = 8\,(x+t)\,e^{-(x+t)^2}, \quad -\infty < x < \infty,\ t > 0.$$

   (a) Introduce an appropriate change of variables to re-write this problem in *the first canonical, or H1, form* for hyperbolic equations.

   (b) Working from the *H1 form* for the above problem, find the solution $u(x,t)$ satisfying the initial conditions

$$u(x,0) = u_t(x,0) = 0.$$

4. Determine the regions in $\mathbb{R}^3$, where the following pde is hyperbolic, elliptic or parabolic:

$$u_{xx} - 2x^2 u_{xz} + u_{yy} + u_{zz} = 0,$$