Solutions for Math 436 2010 Midterm

Question 1: The pde and initial condition are

$$u_t + u\,u_x = 0,\ \ x \in \mathbb{R},\, t > 0,$$

$$u\,(x, 0) = f\,(x) = \begin{cases} 1 \text{ for } x \le 0, \\ 1 - x/a \text{ for } 0 < x \le a, \\ 0 \text{ for } x > a. \end{cases}$$

(a) The initial data curve can be written in the parametric form $x = \tau$ and $t = 0$ with $\tau \in \mathbb{R}$. The characteristic equations are

$$\frac{dt}{ds} = 1 \text{ subject to } \left. t\right|_{s=0} = 0 \Longrightarrow t = s,$$

$$\frac{du}{ds} = 0 \text{ subject to } \left. u\right|_{s=0} = f\,(\tau) \Longrightarrow u = f\,(\tau)\,,$$

$$\frac{dx}{ds} = u = f\,(\tau) \text{ subject to } \left. x\right|_{s=0} = \tau \Longrightarrow x = sf\,(\tau) + \tau.$$

It follows that $u\,(x,t)$ is given by

$$u\,(x,t) = f\,(\tau) \text{ with } \tau = x - tf\,(\tau)\,.$$

We can explicitly determine $u\,(x,t)$ as follows.
For $\tau \le 0$ it follows that $f\,(\tau) = 1$ so that $\tau = x - t$. Therefore

$$\tau \le 0 \Longleftrightarrow x - t \le 0 \Longleftrightarrow x \le t.$$

Hence we conclude that

$$u\,(x,t) = 1 \text{ for } x \le t.$$

For $0 < \tau \le a$ it follows that $f\,(\tau) = 1 - \tau/a$ so that

$$\tau = x - t\,(1 - \tau/a) \Longleftrightarrow \tau = \frac{a\,(x-t)}{a-t}.$$

Therefore

$$0 < \tau \le a \Longleftrightarrow 0 < \frac{a\,(x-t)}{a-t} \le a \Longleftrightarrow 0 < x - t \le a - t \Longleftrightarrow t < x \le a.$$

Hence we conclude that

$$u\,(x,t) = 1 - \tau/a = 1 - \frac{(x-t)}{a-t} = \frac{a-x}{a-t} \text{ for } t < x \le a.$$

For $\tau > a$ it follows that $f\,(\tau) = 0$ so that $\tau = x$. Hence we conclude that

$$u\,(x,t) = 0 \text{ for } x > a.$$

In summary we can write

$$u(x,t) = \begin{cases} 1 \text{ for } x \le t, \\ \frac{a-x}{a-t} \text{ for } t < x \le a, \\ 0 \text{ for } x > a. \end{cases}$$

(b) A shock forms the first time $|u_x| \to \infty$. We have

$$u_x(x,t) = \begin{cases} 0 \text{ for } x < t, \\ -\frac{1}{a-t} \text{ for } t < x < a, \\ 0 \text{ for } x > a. \end{cases}$$

Hence we see that $|u_x| \to \infty$ when $t = a$ and is located at $x = a$. Thus $(x_s, t_s) = (a, a)$.
(c) Let $x = h(t)$ be the position of the shock. The entropy condition states that $h(t)$ is the solution of

$$\frac{dh}{dt} = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \text{ subject to } h(t_s) = x_s,$$

where $f(u)$ is the flux function associated with the pde and $u^+ = \lim_{x \downarrow h} u$ and $u^- = \lim_{x \uparrow h} u$ at the moment of shock formation. In our case it follows that $f(u) = u^2/2$ and $u^+ = 0$ and $u^- = 1$. Hence

$$\frac{dh}{dt} = \frac{\frac{1}{2}\left[(u^+)^2 - (u^-)^2\right]}{u^+ - u^-} = \frac{1}{2}\left(u^+ + u^-\right) = \frac{1}{2} \text{ subject to } h(a) = a.$$

which implies that

$$h(t) = \frac{t+a}{2}.$$

(d) Therefore the solution for $u(x,t)$ when $t \ge a$ is given by

$$u(x,t) = \begin{cases} 1 \text{ for } x \le \frac{t+a}{2}, \\ 0 \text{ for } x > \frac{t+a}{2}. \end{cases} .$$

Question 2: The pde and initial condition, given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y),$$

$$u(x, h(x)) = f(x),$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x,h)}{a(x,h)}.$$

It follows that

$$\frac{df}{dx} = u_x(x, h(x)) + u_y(x, h(x))\,\frac{dh}{dx}$$

$$= u_x(x,h(x)) + \frac{b(x,h)}{a(x,h)} u_y(x,h(x)) = \frac{a(x,h)\, u_x(x,h(x)) + b(x,h)\, u_y(x,h(x))}{a(x,h)}$$

$$= \frac{c(x,h)\, u(x,h(x)) + d(x,h)}{a(x,h)} = \frac{c(x,h)\, f + d(x,h)}{a(x,h)}.$$

Question 3: The pde is given by

$$u_{tt} - u_{xx} = 8(x+t)\, e^{-(x+t)^2}, \quad -\infty < x < \infty,\ t > 0,$$

(a) The $\omega(x,y)$ functions are determined by

$$\omega^{\pm} = \pm 1.$$

To reduce to *H1* canonical form in the hyperbolic case, we introduce the characteristic variables (ξ, η) as determined by

$$\left(\frac{dx}{dt}\right)_{\xi} = -\omega^{+} = -1 \Longrightarrow \xi = x + t,$$

$$\left(\frac{dx}{dt}\right)_{\eta} = -\omega^{-} = 1 \Longrightarrow \eta = x - t,$$

with the "inverse" relations

$$x = \frac{\xi + \eta}{2} \text{ and } t = \frac{\xi - \eta}{2}.$$

It follows, therefore, that

$$u_x = u_\xi + u_\eta \Longrightarrow u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$$u_t = u_\xi - u_\eta \Longrightarrow u_{tt} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.$$

Substitution into the pde yields

$$u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = -4u_{\xi\eta} = 8(x+t)\, e^{-(x+t)^2} = 8\xi e^{-\xi^2}$$

$$\Longrightarrow u_{\xi\eta} = -2\xi e^{-\xi^2}.$$

(b) The pde just given may be integrated with respect to ξ to give

$$u_\eta = \phi(\eta) + e^{-\xi^2},$$

where ϕ is an arbitrary function of its argument. We may integrate again with respect to η to find

$$u = \Phi(\eta) + \Psi(\xi) + \eta e^{-\xi^2},$$

where Φ and Ψ are arbitrary functions of their arguments. Written in terms of (x,t) the solution is therefore given by

$$u(x,t) = \Phi(x-t) + \Psi(x+t) + (x-t)\, e^{-(x+t)^2}.$$

Now, application of the initial condition $u(x,0) = 0$ leads to

$$\Phi(x) + \Psi(x) = -xe^{-x^2}.$$

And application of the initial condition $u_t(x,0) = 0$ leads to

$$-\Phi'(x) + \Psi'(x) = e^{-x^2} + 2x^2e^{-x^2} = \left(1+2x^2\right)e^{-x^2}.$$

Thus

$$\left[xe^{-x^2} + \Psi(x)\right]' + \Psi'(x) = \left(1+2x^2\right)e^{-x^2}$$
$$\Longrightarrow \Psi'(x) = 2x^2e^{-x^2} \Longrightarrow \Psi(x) = 2\int_0^x s^2e^{-s^2}\,ds$$
$$\Longrightarrow \Phi(x) = -xe^{-x^2} - 2\int_0^x s^2e^{-s^2}\,ds.$$

Hence

$$u(x,t) = (x-t)\,e^{-(x+t)^2} - (x-t)\,e^{-(x-t)^2} - 2\int_0^{x-t} s^2e^{-s^2}\,ds + 2\int_0^{x+t} s^2e^{-s^2}\,ds$$
$$= (x-t)\left[e^{-(x+t)^2} - e^{-(x-t)^2}\right] + 2\int_{x-t}^{x+t} s^2e^{-s^2}\,ds$$
$$= (x-t)\left[e^{-(x+t)^2} - e^{-(x-t)^2}\right] - \int_{x-t}^{x+t} s\,\frac{de^{-s^2}}{ds}\,ds$$
$$= (x-t)\left[e^{-(x+t)^2} - e^{-(x-t)^2}\right] - \left[se^{-s^2}\right]_{x-t}^{x+t} + \int_{x-t}^{x+t} e^{-s^2}\,ds$$
$$= -2te^{-(x+t)^2} + \int_{x-t}^{x+t} e^{-s^2}\,ds.$$

Question 4: The pde can be written in the matrix form

$$\begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} A \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} u = 0 \text{ where } A = \begin{bmatrix} 1 & 0 & -x^2 \\ 0 & 1 & 0 \\ -x^2 & 0 & 1 \end{bmatrix}.$$

The eigenvalues $\{\lambda_i\}_{i=1}^3$ of A are given by

$$|A-\lambda I| = \begin{bmatrix} 1-\lambda & 0 & -x^2 \\ 0 & 1-\lambda & 0 \\ -x^2 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\left[(1-\lambda)^2 - x^4\right] = 0$$
$$\Longrightarrow \lambda_1 = 1 > 0,\ \lambda_2 = 1+x^2 > 0 \text{ and } \lambda_3 = 1-x^2.$$

Hence, we conclude that

If $|x| < 1$, the pde is *elliptic*,

If $|x| = 1$, the pde is *parabolic*,

If $|x| > 1$, the pde is *hyperbolic*.