

Solutions for Math 436 2010 Final

Question 1a: Let $f(x)$ and $g(x)$ be smooth square-integrable functions that satisfy the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be self-adjoint if

$$(f, \mathcal{L}g) = (g, \mathcal{L}f).$$

To show $\frac{1}{\rho}L$ is self-adjoint, we will show that

$$\left(f, \frac{1}{\rho}Lg\right) - \left(g, \frac{1}{\rho}Lf\right) = 0.$$

We have

$$\begin{aligned} \left(f, \frac{1}{\rho}Lg\right) - \left(g, \frac{1}{\rho}Lf\right) &= \int_G f Lg - g Lf \, dx \\ &= \int_G f [-\nabla \cdot (p\nabla g) + qg] - g [-\nabla \cdot (p\nabla f) + qf] \, dx \\ &= \int_G g \nabla \cdot (p\nabla f) - f \nabla \cdot (p\nabla g) \, dx \\ &= \int_{\partial G} g \mathbf{n} \cdot (p\nabla f) - f \mathbf{n} \cdot (p\nabla g) \, dx + \int_G p [\nabla f \cdot \nabla g - \nabla g \cdot \nabla f] \, dx \\ &= \int_{\partial G} p \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) \, dx. \end{aligned}$$

If $\beta = 0$, then $f = g = 0$ for $x \in \partial G$ so that

$$\int_{\partial G} p \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) \, dx = 0.$$

If $\beta \neq 0$, then

$$\begin{aligned} \frac{\partial f}{\partial n} &= -\frac{\alpha f}{\beta} \text{ and } \frac{\partial g}{\partial n} = -\frac{\alpha g}{\beta} \text{ for } x \in \partial G \\ \implies \int_{\partial G} p \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) \, dx &= \int_{\partial G} p \left(\frac{\alpha g f}{\beta} - \frac{\alpha g f}{\beta} \right) \, dx = 0. \end{aligned}$$

Thus we have shown that

$$\left(f, \frac{1}{\rho}Lg\right) = \left(g, \frac{1}{\rho}Lf\right).$$

Question 1b: Let $f(x)$ be a smooth square-integrable function that satisfies the boundary condition associated with the differential operator $\mathcal{L}$, then $\mathcal{L}$ is said to be positive if

$$(f, \mathcal{L}f) \geq 0.$$

We proceed directly:

$$\begin{aligned} \left(f, \frac{1}{\rho} Lf \right) &= \int_G f Lf \, dx = \int_G f [-\nabla \cdot (p \nabla f) + qf] \, dx \\ &= - \int_{\partial G} p f \frac{\partial f}{\partial n} \, dx + \int_G p \nabla f \cdot \nabla f + q f^2 \, dx. \end{aligned}$$

If $\beta = 0$, then $f = 0$ for $x \in \partial G$ so that

$$\left(f, \frac{1}{\rho} Lf \right) = \int_G p \nabla f \cdot \nabla f + q f^2 \, dx \geq 0,$$

since $p > 0$ and $q \geq 0$. If $\beta \neq 0$, then

$$\begin{aligned} \frac{\partial f}{\partial n} &= -\frac{\alpha f}{\beta} \text{ for } x \in \partial G \\ \implies \left(f, \frac{1}{\rho} Lf \right) &= \int_{\partial G} \frac{\alpha p f^2}{\beta} \, dx + \int_G p \nabla f \cdot \nabla f + q f^2 \, dx \geq 0, \end{aligned}$$

since $p > 0$, $q \geq 0$, $\alpha \geq 0$ and $\beta > 0$.

Question 1c: The eigenvalue problem is given by

$$\frac{1}{\rho} Lu = \lambda u, \, x \in G,$$

with the boundary condition

$$\alpha(x) u + \beta(x) \frac{\partial u}{\partial n} = 0 \text{ for } x \in \partial G.$$

Since $\frac{1}{\rho} L$ is a positive operator

$$0 \leq \left(u, \frac{1}{\rho} Lu \right) = \lambda (u, u) = \lambda \|u\|^2 \implies \lambda \geq 0.$$

Question 2a: The Fourier Series is defined as

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k) \varphi_k(x).$$

Question 2b: The n^{th} partial sum is given by

$$\psi_n(x) = \sum_{k=1}^n (\varphi, \varphi_k) \varphi_k(x).$$

To show Bessel's Inequality, we begin with

$$\begin{aligned}
0 &\leq \|\varphi(x) - \psi_n(x)\|^2 = (\varphi - \psi_n, \varphi - \psi_n) = (\varphi, \varphi) - 2(\varphi, \psi_n) + (\psi_n, \psi_n) \\
&= (\varphi, \varphi) - 2 \left(\varphi, \sum_{k=1}^n (\varphi, \varphi_k) \varphi_k(x) \right) + \left(\sum_{m=1}^n (\varphi, \varphi_m) \varphi_m(x), \sum_{k=1}^n (\varphi, \varphi_k) \varphi_k(x) \right) \\
&= (\varphi, \varphi) - 2 \sum_{k=1}^n (\varphi, \varphi_k) (\varphi, \varphi_k) + \sum_{m=1}^n \sum_{k=1}^n (\varphi, \varphi_k) (\varphi, \varphi_m) (\varphi_m, \varphi_k) \\
&= (\varphi, \varphi) - 2 \sum_{k=1}^n (\varphi, \varphi_k)^2 + \sum_{k=1}^n (\varphi, \varphi_k)^2 = (\varphi, \varphi) - \sum_{k=1}^n (\varphi, \varphi_k)^2 \\
&\implies \sum_{k=1}^n (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).
\end{aligned}$$

Since the right-hand-side of this expression is independent of n , this inequality must hold for all n regardless of large it is, and thus in the limit $n \rightarrow \infty$, it follows

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).$$

Question 2c: Mean square convergence is defined as

$$\lim_{n \rightarrow \infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

Question 2d: We must show that

$$\lim_{n \rightarrow \infty} \|\varphi(x) - \psi_n(x)\| = 0 \iff \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi).$$

From Question 2b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^n (\varphi, \varphi_k)^2.$$

Thus, provided the limit exists,

$$\lim_{n \rightarrow \infty} \|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \lim_{n \rightarrow \infty} \sum_{k=1}^n (\varphi, \varphi_k)^2.$$

Hence

$$\lim_{n \rightarrow \infty} \|\varphi(x) - \psi_n(x)\|^2 = 0 \implies \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi),$$

and

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) \implies \lim_{n \rightarrow \infty} \|\varphi(x) - \psi_n(x)\|^2 = 0.$$

Question 3: Assume two solutions exist to the problem denoted by $u_1(x, t)$ and $u_2(x, t)$, respectively, i.e.,

$$\begin{aligned} \partial_{tt}u_1 + \mathcal{L}u_1 &= F(x, t), \quad x \in G, \quad t > 0, \\ u_1(x, 0) &= f(x), \quad \partial_t u_1(x, 0) = g(x) \quad \text{for } x \in G, \\ \text{and } \alpha u_1 + \beta \frac{\partial u_1}{\partial n} &= B(x, t) \quad \text{for } x \in \partial G, \quad t > 0, \end{aligned}$$

and

$$\begin{aligned} \partial_{tt}u_2 + \mathcal{L}u_2 &= F(x, t), \quad x \in G, \quad t > 0, \\ u_2(x, 0) &= f(x), \quad \partial_t u_2(x, 0) = g(x) \quad \text{for } x \in G, \\ \text{and } \alpha u_2 + \beta \frac{\partial u_2}{\partial n} &= B(x, t) \quad \text{for } x \in \partial G, \quad t > 0. \end{aligned}$$

Define the difference $w = u_1 - u_2$, it follows that w satisfies

$$\begin{aligned} w_{tt} + \mathcal{L}w &= 0, \quad x \in G, \quad t > 0, \\ w(x, 0) &= 0, \quad w_t(x, 0) = 0 \quad \text{for } x \in G, \\ \text{and } \alpha w + \beta \frac{\partial w}{\partial n} &= 0 \quad \text{for } x \in \partial G, \quad t > 0. \end{aligned}$$

We form the energy equation

$$w_t w_{tt} + w_t \mathcal{L}w = 0 \implies \partial_t \|w_t\|^2 + 2(w_t, \mathcal{L}w) = 0,$$

and since $\mathcal{L}$ is a *self-adjoint operator* it follows that $(w_t, \mathcal{L}w) = (w, \mathcal{L}w_t)$, so that the energy equation can be written in the form

$$\begin{aligned} \partial_t \|w_t\|^2 + (w_t, \mathcal{L}w) + (w, \mathcal{L}w_t) &= 0 \implies \frac{\partial}{\partial t} \left[\|w_t\|^2 + (w, \mathcal{L}w) \right] = 0 \\ \implies \|w_t\|^2 + (w, \mathcal{L}w) &= \left[\|w_t\|^2 + (w, \mathcal{L}w) \right]_{t=0} = 0 \\ \implies \|w_t\|^2 &= -(w, \mathcal{L}w) \leq 0, \end{aligned}$$

since $\mathcal{L}$ is a *positive operator*, i.e., $(w, \mathcal{L}w) \geq 0$. Hence

$$\|w_t\| = 0 \implies w_t = 0 \implies w(x, t) = w(x, 0) = 0.$$

Question 4: The eigenvalue problem is given by

$$-u_{xx} = \lambda^2 u \quad \text{for } 0 < x < 1,$$

subject to

$$u(0) = u(1) = 0.$$

The general solution is given by

$$u = A \cos (\lambda x) + B \sin (\lambda x) .$$

Application of the boundary conditions leads to

$$u (0) = 0 = A,$$

$$u (1) = 0 = B \sin (\lambda) \implies \lambda = n\pi \text{ for } n = 1, 2, 3, \dots$$

Thus, the orthonormalized eigenfunctions are given by

$$u_n (x) = \frac{\sin (n\pi x)}{\left[\int_0^1 \sin^2 (n\pi x) \, dx \right]^{\frac{1}{2}}},$$

where

$$\int_0^1 \sin^2 (n\pi x) \, dx = \int_0^1 \frac{1 - \cos (2n\pi x)}{2} \, dx = \frac{1}{2}.$$

Hence

$$u_n (x) = \sqrt{2} \sin (n\pi x) .$$