Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

28 October 2008

Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Using the Method of Characteristics, show that the solution $u(x,t)$ for

$$u_t + u\,u_x = -cu, \quad -\infty < x < \infty,\ t > 0,\ c > 0,$$

$$u(x,0) = ax, \quad -\infty < x < \infty,\ a \in \mathbb{R},$$

is given by

$$u(x,t) = \frac{acx}{(a+c)\exp(ct) - a}.$$

Show that a shock will form *only* if $a < -c$. Assuming $a < -c$, determine the space-time coordinates (x_s, t_s) of first shock formation.

2. Suppose that the initial data for the linear 1^{st}-order pde

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y),$$

given by

$$u(x, h(x)) = f(x),$$

where $y = h(x)$ is a characteristic, and a, b, c and d are smooth functions. Show that it is impossible to uniquely determine *both* $u_x(x, h(x))$ and $u_y(x, h(x))$, if a solution exists at all.

3. Determine the regions in $\mathbb{R}^2$ where the linear 2^{nd}-order pde

$$u_{xx} + y u_{yy} + \frac{1}{2} u_y = 0,$$

is hyperbolic, elliptic or parabolic. For *only the hyperbolic case*, transform the pde into *H1* canonical form and determine the general solution in terms of (x,y) variables.

4. Show that the solution to the Cauchy problem

$$u_{tt} - u_{xx} = h(x,t), \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x), \quad -\infty < x < \infty,$$

where $h(x,t)$, $f(x)$ and $g(x)$ are smooth and spatially square-integrable functions is unique. HINT: You may assume that $u(x,t)$ and all its derivatives are square-integrable functions with respect to x.