Solutions for Math 436 2008 Midterm

Question 1: The pde and initial condition are

$$u_t + uu_x = -cu, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = ax, \quad -\infty < x < \infty,$$

where $c > 0$. The initial data curve can be written in the parametric form $x = \tau$ and $t = 0$ with $\tau \in \mathbb{R}$. The characteristic equations are

$$\frac{dt}{ds} = 1 \text{ subject to } \left.t\right|_{s=0} = 0 \Longrightarrow t = s, \tag{1}$$

$$\frac{du}{ds} = -cu \text{ subject to } \left.u\right|_{s=0} = a\tau \Longrightarrow u = a\tau\exp(-cs), \tag{2}$$

$$\frac{dx}{ds} = u = a\tau\exp(-cs) \text{ subject to } \left.x\right|_{s=0} = \tau$$

$$\Longrightarrow x = \tau\left\{1 + \frac{a[1-\exp(-cs)]}{c}\right\}. \tag{3}$$

It follows from (1) and (3) that the "inverse relations" are

$$s = t \text{ and } \tau = \frac{cx}{c + a[1-\exp(-ct)]},$$

so that $u(x,t)$ is given by

$$u(x,t) = \frac{acx}{(c+a)\exp(ct) - a}.$$

A shock will form the first time

$$|u_x| = \left|\frac{ac}{(c+a)\exp(ct) - a}\right| \to \infty,$$

i.e., when

$$\exp(-ct) = \frac{a+c}{a} = 1 + \frac{c}{a}.$$

Note that this is independent of x. But $0 \leq \exp(-ct) \leq 1$ since $c > 0$ and $t \geq 0$, so that a shock will form only if $-1 \leq c/a \leq 0$. And since $c > 0$ this means that a shock will form only if $a < -c$ and a shock will never form if $a \geq -c$. If $a < -c$, the shock will form at the time $t = t_s$ given by

$$\exp(-ct_s) = \frac{a+c}{a} \Longrightarrow t_s = \frac{1}{c}\ln\left(\frac{a}{a+c}\right) > 0.$$

And the shock forms everywhere, i.e., for all $-\infty < x < \infty$.

Question 2: The pde and initial condition, given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)\,,$$

$$u(x,h(x)) = f(x)\,,$$

where $y = h(x)$ is a characteristic, where a, b, c and d are smooth functions. Since $y = h(x)$ is a characteristic, it follows that

$$\frac{dh(x)}{dx} = \frac{b(x,h)}{a(x,h)}. \tag{4}$$

It follows that

$$u_x(x,h(x)) + u_y(x,h(x))\,\frac{dh}{dx} = f'(x)\,,$$

and

$$a(x,h)\,u_x(x,h(x)) + b(x,h)\,u_y(x,h(x)) = c(x,h)\,u(x,h(x)) + d(x,h)\,,$$

which can be written in the matrix form

$$\left[\begin{array}{cc} 1 & \frac{dh(x)}{dx} \\ a(x,h) & b(x,h) \end{array}\right]\left[\begin{array}{c} u_x(x,h(x)) \\ u_y(x,h(x)) \end{array}\right] = \left[\begin{array}{c} f'(x) \\ c(x,h)\,u(x,h(x)) + d(x,h) \end{array}\right]. \tag{5}$$

But

$$\det\left[\begin{array}{cc} 1 & \frac{dh(x)}{dx} \\ a(x,h) & b(x,h) \end{array}\right] = b(x,h) - a(x,h)\,\frac{dh(x)}{dx} = 0,$$

on account of (4). Thus (5) cannot be inverted to uniquely determine $u_x(x,h(x))$ and $u_y(x,h(x))$.

Question 3: The pde is given by

$$u_{xx} + y u_{yy} + \frac{1}{2}u_y = 0. \tag{6}$$

The $\omega(x,y)$ functions are determined by

$$\omega^2 + y = 0 \Longleftrightarrow \omega^{\pm} = \pm\sqrt{-y}.$$

Thus for the classification, we find

If $y > 0$, the pde is *elliptic*.

If $y = 0$, the pde is *parabolic*.

If $y < 0$, the pde is *hyperbolic*.

To reduce to *H1* canonical form in the hyperbolic case, we introduce the characteristic variables (ξ, η) as determined by

$$\left(\frac{dy}{dx}\right)_{\xi} = -\omega^{+} = -\sqrt{-y} \Longrightarrow \xi = x - 2\sqrt{-y},$$

$$\left(\frac{dy}{dx}\right)_\eta = -\omega^- = \sqrt{-y} \Longrightarrow \eta = x + 2\sqrt{-y},$$

with the "inverse" relations

$$x = \frac{\xi + \eta}{2} \text{ and } y = -\frac{(\eta - \xi)^2}{16}.$$

It follows, therefore, that

$$u_x = u_\xi + u_\eta \Longrightarrow u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$$u_y = \frac{1}{\sqrt{-y}}\left(u_\xi - u_\eta\right) \Longrightarrow u_{yy} = \frac{1}{2\left(-y\right)^{\frac{3}{2}}}\left(u_\xi - u_\eta\right) - \frac{1}{y}\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right).$$

Substitution into (6) yields

$$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} + y\left[\frac{1}{2\left(-y\right)^{\frac{3}{2}}}\left(u_\xi - u_\eta\right) - \frac{1}{y}\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right)\right]$$
$$+\frac{1}{2}\left[\frac{1}{\sqrt{-y}}\left(u_\xi - u_\eta\right)\right] = 4u_{\xi\eta} = 0. \tag{7}$$

The pde (7) has the general solution

$$u = \Phi\left(\xi\right) + \Psi\left(\eta\right),$$

where Φ and Ψ are arbitrary functions of their arguments. Thus, in terms of x and y, the general solution to (6) is given by

$$u\left(x, y\right) = \Phi\left(x - 2\sqrt{-y}\right) + \Psi\left(x + 2\sqrt{-y}\right),$$

when $y < 0$.

Question 4: The pde and initial conditions are given by

$$u_{tt} - u_{xx} = h\left(x, t\right), \quad -\infty < x < \infty,\ t > 0,$$

$$u\left(x, 0\right) = f\left(x\right) \text{ and } u_t\left(x, 0\right) = g\left(x\right), \quad -\infty < x < \infty,$$

where $h\left(x, t\right)$, $f\left(x\right)$ and $g\left(x\right)$ are smooth and spatially square-integrable functions is unique. To show uniqueness, we assume that there are two solutions, given by $u_1\left(x, t\right)$ and $u_2\left(x, t\right)$, i.e.,

$$\left(\partial_{tt} - \partial_{xx}\right) u_1 = h, \quad -\infty < x < \infty,\ t > 0,$$

$$u_1\left(x, 0\right) = f \text{ and } \partial_t u_1\left(x, 0\right) = g, \quad -\infty < x < \infty,$$

and

$$\left(\partial_{tt} - \partial_{xx}\right) u_2 = h, \quad -\infty < x < \infty,\ t > 0,$$
$$u_2\left(x, 0\right) = f \text{ and } \partial_t u_2\left(x, 0\right) = g, \quad -\infty < x < \infty.$$

Let $\Phi(x,t) = u_1(x,t) - u_2(x,t)$. We will show that $\Phi(x,t) = 0$ for all $t \geq 0$. Hence $u_1(x,t) = u_2(x,t)$ and we have established uniqueness. It follows that

$$(\partial_{tt} - \partial_{xx})\,\Phi = 0, \quad -\infty < x < \infty,\ t > 0, \tag{8}$$

$$\Phi(x,0) = 0 \text{ and } \Phi_t(x,0) = 0, \quad -\infty < x < \infty. \tag{9}$$

The energy equation associated is obtained by multiplying (8) by Φ_t and rewriting the resulting equation as a space-time divergence, i.e.,

$$\Phi_t\,(\partial_{tt} - \partial_{xx})\,\Phi = \frac{1}{2}\partial_t\left[(\Phi_t)^2 + (\Phi_x)^2\right] - \partial_x(\Phi_t\Phi_x) = 0.$$

It therefore follows that

$$\partial_t \int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2\ dx = 2\Phi_t\Phi_x|_{-\infty}^{\infty} = 0, \tag{10}$$

since $\Phi_t\Phi_x \to 0$ as $|x| \to \infty$ since $\Phi_{x,t}$ are smooth square-integrable functions. Thus, it follows from (10) that

$$\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2\ dx = \left[\int_{-\infty}^{\infty} (\Phi_t)^2 + (\Phi_x)^2\ dx\right]_{t=0} = 0, \tag{11}$$

since $\Phi(x,0) = 0$ ($\Longrightarrow \Phi_x(x,0) = 0$) and $\Phi_t(x,0) = 0$. Further, it then follows from (11) that

$$\Phi_t(x,t) = \Phi_x(x,t) = 0 \text{ for all } t \geq 0 \Longrightarrow \Phi(x,t) = 0 \text{ for all } t \geq 0.$$