Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

10 December 2008

Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is an orthonormal sequence of square-integrable functions defined on $x \in G \subset \mathbb{R}^n$ with the inner product

$$(u, w) \equiv \int_G \rho\, u\, w\; dx, \text{ with } \rho = \rho(x) > 0.$$

   (a) If $\varphi(x)$ is a square-integrable function for $x \in G$, define the *Fourier Series* for $\varphi(x)$ with respect to $\{\varphi_k(x)\}_{k=1}^{\infty}$.

   (b) Beginning with the n^{th} partial sum associated with the Fourier Series for $\varphi(x)$, denoted by $\psi_n(x)$, show that *Bessel's Inequality* holds, i.e.,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2 .$$

   (c) Define what it means for a sequence of square-integrable functions $\{\psi_n(x)\}_{n=1}^{\infty}$ to *converge* to a function $\varphi(x)$ *in the mean.*

   (d) Show that *convergence in the mean* is equivalent to *Parseval's Identity.*

2. Let $\mathcal{L}$ be a *positive, self-adjoint, real-valued partial differential operator* defined for smooth square-integrable functions $f(x)$ where $x \in G \subset \mathbb{R}^n$ and satisfying the boundary conditions

$$\alpha f + \beta \frac{\partial f}{\partial n} = 0, \text{ with } \alpha, \beta \geq 0 \text{ where } \alpha + \beta > 0, \text{ for } x \in \partial G.$$

   (a) Define what it means for $\mathcal{L}$ to be a *positive, self-adjoint operator* with respect to the inner product (f, g).

   (b) Let M and λ be an Eigenfunction and corresponding Eigenvalue for the operator $\mathcal{L}$ (with unit weight function).

      1. Show that if M and λ is an Eigenfunction and corresponding Eigenvalue for the operator $\mathcal{L}$, so is the complex conjugate of M and λ.
      2. Show that all the Eigenvalues are real.
      3. Show that all the Eigenvalues are non-negative.

3. Consider the following hyperbolic Cauchy problem:

$$u_{tt} - u_{xx} = 8\,(x+t)\,e^{-(x+t)^2}, \quad -\infty < x < \infty,\ t > 0,$$

where is it assumed that $u\,(x,t)$ and all its derivative are square-integrable functions for $x \in \mathbb{R}$ for all $t \geq 0$.

(a) Introduce an appropriate change of variables to re-write this problem in *the first canonical, or H1, form* for hyperbolic equations.

(b) Working from *the H1 canonical form* for the above problem, show that the general solution is given by

$$u\,(x,t) = \Psi\,(x+t) + \Phi\,(x-t) + (x-t)\,e^{-(x+t)^2},$$

where Ψ and Φ are arbitrary functions of their arguments.

(c) Working from the result in Part b, show that the solution $u\,(x,t)$ satisfying the initial conditions

$$u\,(x,0) = u_t\,(x,0) = 0,$$

is given by

$$u\,(x,t) = -2te^{-(x+t)^2} + \int_{x-t}^{x+t} e^{-\zeta^2}\,d\zeta.$$

4. Determine the stability index Ω for the pde

$$u_t - u_{xx} - u_x + au = 0,$$

as a function of the parameter a. Hence, determine the stability of the pde as a function of the parameter a.