<u>Solutions for Math 436 2008 Final</u>

Question 1a: The Fourier Series is defined as

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k)\ \varphi_k(x).$$

Question 1b: The n^{th} partial sum is given by

$$\psi_n(x) = \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x).$$

To show Bessel's Inequality, we begin with

$$0 \le \|\varphi(x) - \psi_n(x)\|^2 = (\varphi - \psi_n, \varphi - \psi_n) = (\varphi, \varphi) - 2(\varphi, \psi_n) + (\psi_n, \psi_n)$$

$$= (\varphi, \varphi) - 2\left(\varphi, \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right) + \left(\sum_{m=1}^{n} (\varphi, \varphi_m)\ \varphi_m(x), \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi,\ \varphi_k) + \sum_{m=1}^{n}\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_m)(\varphi_m,\ \varphi_k)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)^2 + \sum_{k=1}^{n} (\varphi, \varphi_k)^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2$$

$$\Longrightarrow \sum_{k=1}^{n} (\varphi, \varphi_k)^2 \le (\varphi, \varphi).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit $n \to \infty$, it follows

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \le (\varphi, \varphi).$$

Question 1c: Mean square convergence is defined as

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

Question 1d: We must show that

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi).$$

From Question 1b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Thus, provided the limit exists,

$$\lim_{n\to\infty}\|\varphi(x)-\psi_n(x)\|^2=(\varphi,\varphi)-\lim_{n\to\infty}\sum_{k=1}^{n}(\varphi,\varphi_k)^2.$$

Hence

$$\lim_{n\to\infty}\|\varphi(x)-\psi_n(x)\|^2=0\Longrightarrow\sum_{k=1}^{\infty}(\varphi,\varphi_k)^2=(\varphi,\varphi),$$

and

$$\sum_{k=1}^{\infty}(\varphi,\varphi_k)^2=(\varphi,\varphi)\Longrightarrow\lim_{n\to\infty}\|\varphi(x)-\psi_n(x)\|^2=0.$$

Question 2a: The operator $\mathcal{L}$ is positive and self-adjoint if for all $f(x)$ and $g(x)$ that are in $L_2(G)$ and satisfy the boundary conditions, it follows that

$$(f,\mathcal{L}f)\geq 0 \text{ and } (f,\mathcal{L}g)=(g,\mathcal{L}f),$$

respectively.
Question 2b(1): The Eigenvalue problem is given by

$$\mathcal{L}M=\lambda M \text{ for } x\in G,$$

$$\alpha M+\beta\frac{\partial M}{\partial n}=0 \text{ for } x\in\partial G.$$

It follows, since α, β, x and $\mathcal{L}$ are real-valued, that

$$\mathcal{L}M^*=\lambda^*M^* \text{ for } x\in G,$$

$$\alpha M^*+\beta\frac{\partial M^*}{\partial n}=0 \text{ for } x\in\partial G,$$

where the $*$ superscript is the complex-conjugate, i.e., M^* and λ^* solves the Eigenvalue problem if M and λ does.
Question 2b(2): To show that $\lambda\in\mathbb{R}$, we exploit the self-adjointness of $\mathcal{L}$ as follows:

$$(M^*,\mathcal{L}M)=(\mathcal{L}M^*,M)\Longleftrightarrow\lambda(M^*,M)=\lambda^*(M^*,M)$$

$$\Longleftrightarrow(\lambda-\lambda^*)\int_G|M|^2\,dx=0\Longrightarrow\lambda=\lambda^* \text{ since } \int_G|M|^2\,dx>0.$$

Question 2b(3): To show that $\lambda\geq 0$, we exploit the positiveness of $\mathcal{L}$ as follows:

$$(M,\mathcal{L}M)=\lambda(M,M)=\lambda\|M\|^2\Longrightarrow\lambda=\frac{(M,\mathcal{L}M)}{\|M\|^2}\geq 0.$$

Question 3a: Introduce the characteristic variables

$$\xi=x+t \text{ and } \eta=x-t.$$

It follows that

$$\partial_t = \partial_\xi - \partial_\eta \Longrightarrow \partial_{tt} = \partial_{\xi\xi} - 2\partial_{\eta\xi} + \partial_{\eta\eta},$$
$$\partial_x = \partial_\xi + \partial_\eta \Longrightarrow \partial_{xx} = \partial_{\xi\xi} + 2\partial_{\eta\xi} + \partial_{\eta\eta},$$

so that the pde maps to

$$u_{\xi\eta} = -2\xi e^{-\xi^2}.$$

Question 3b: The general solution can be obtained as follows. First, we integrate with respect to ξ and get

$$u_\eta = \phi(\eta) + e^{-\xi^2},$$

where ϕ is an arbitrary function of its argument. Then, we integrate with respect to η, and obtain

$$u = \Psi(\xi) + \Phi(\eta) + \eta e^{-\xi^2},$$

where Ψ and Φ are arbitrary functions of their arguments. Thus, finally we have

$$u(x,t) = \Psi(x+t) + \Phi(x-t) + (x-t)\, e^{-(x+t)^2}.$$

Question 3c: We have

$$u(x,0) = \Psi(x) + \Phi(x) + xe^{-x^2} = 0 \Longrightarrow \Phi(x) = -\Psi(x) - xe^{-x^2},$$
$$u_t(x,0) = \Psi'(x) - \Phi'(x) - \left(1+2x^2\right)e^{-x^2} = 0.$$
$$\Longrightarrow \Psi'(x) + \left[\Psi(x) + xe^{-x^2}\right]' = \left(1+2x^2\right)e^{-x^2}$$
$$\Longrightarrow \Psi'(x) = 2x^2 e^{-x^2} \Longrightarrow \Psi(x) = -xe^{-x^2} + \int_0^x e^{-\zeta^2} d\zeta$$
$$\Longrightarrow \Phi(x) = -\int_0^x e^{-\zeta^2} d\zeta = \int_x^0 e^{-\zeta^2} d\zeta.$$
$$\Longrightarrow u(x,t) = -(x+t)\, e^{-(x+t)^2} + (x-t)\, e^{-(x+t)^2} + \int_{x-t}^{x+t} e^{-\zeta^2} d\zeta$$
$$= -2te^{-(x+t)^2} + \int_{x-t}^{x+t} e^{-\zeta^2} d\zeta.$$

Alternatively, using the Method of Characteristics, we have shown in class that

$$u(x,t) = 4\int_0^t \int_{x-t+\tau}^{x+t-\tau} (\sigma+\tau)\, e^{-(\sigma+\tau)^2}\, d\sigma d\tau = -2\int_0^t e^{-(\sigma+\tau)^2}\Big|_{x-t+\tau}^{x+t-\tau} d\tau$$
$$= 2\int_0^t e^{-(x-t+2\tau)^2} - e^{-(x+t)^2}\, d\tau = -2te^{-(x+t)^2} + \int_{x-t}^{x+t} e^{-\zeta^2} d\zeta.$$

Question 4: The pde is

$$u_t - u_{xx} - u_x + au = 0.$$

To compute the stability index we assume a plane wave solution in the form

$$u = A \exp(ikx + \lambda t) + c.c..$$

Substitution into the pde yields

$$\lambda = -k^2 - a + ik \Longrightarrow \mathrm{Re}(\lambda) = -k^2 - a.$$

Thus,

$$\Omega = lub_k [\mathrm{Re}(\lambda)] = -a.$$

Hence, if $a > 0$, the pde is *strictly stable*, if $a = 0$, the pde is *neutrally stable* and if $a < 0$, the pde is *unstable*.