Mathematics 436 Midterm Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Using the method of characteristics, find the solution $u(x,t)$ for

$$u_t + u\, u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) \equiv \begin{cases} 1 - x^2, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Determine the space-time coordinates (x_s, t_s) and the value of $u(x_s, t_s)$ the first time a shock develops.

2. Determine the solution to the hyperbolic Cauchy problem

$$u_{xx} + 4u_{xy} + 3u_{yy} + 2(u_x + u_y) = -4\exp(x - y)$$

$$u(x,0) = 3\left(x^2 + \sin 3x\right) e^x,$$

$$u_x(x,0) + 3u_y(x,0) = -6x(x+1)\, e^x.$$

by introducing characteristic variables, reducing to *H1* canonical form, solving and finally transforming back to (x,y) variables.

3. Show that

$$\mathbf{u}_y + A\mathbf{u}_x = -\mathbf{u}, \text{ where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix},$$

is a totally or strictly hyperbolic system. Determine the characteristic curves and reduce the system to canonical form. Suppose the initial data is given by

$$\mathbf{u}(x,x) = \mathbf{f}(x) = (\phi(x),\ 1,\ -e^x)^\top.$$

Find $\phi(x)$ satisfying $\phi(0) = 0$ for this to be a *consistent* characteristic initial-value problem.

4. Determine the stability index Ω for the parabolic pde

$$u_t - u_{xx} - u_x + \beta u = 0,$$

as a function of the parameter $\beta \in \mathbb{R}$. Determine the stability of the zero solution as a function of β. Are there any values of β for which the pde is not well-posed in the sense of Hadamard?