Solutions for Math 436 2006 Midterm

Question 1: The pde is

$$u_t + u\,u_x = 0, \quad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = f(x) \equiv \begin{cases} 1 - x^2, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

The initial data curve can be parameterized as $x = \tau \in \mathbb{R}$ and $t = 0$ with the initial data $u = f(\tau)$. The characteristic equations are

$$\frac{dt}{ds} = 1 \text{ subject to } \left. t\right|_{s=0} = 0 \Longrightarrow t = s,$$

$$\frac{du}{ds} = 0 \text{ subject to } \left. u\right|_{s=0} = f(\tau) \Longrightarrow u = f(\tau),$$

$$\frac{dx}{ds} = u = f(\tau) \text{ subject to } \left. x\right|_{s=0} = \tau \Longrightarrow x = \tau + sf(\tau).$$

Thus, we may write the solution in the form

$$u(x,t) = f(\tau(x,t)) \text{ where } \tau(x,t) \text{ is determined from } \tau = x - tf(\tau).$$

For $|\tau| \geq 1$, $f(\tau) = 0$ so that $\tau = x$ (the characteristics in this region are simply all lines parallel to the t-axis) and hence the space-time region corresponding to $|\tau| \geq 1$ is $|x| \geq 1\ \forall\ t \in [0, t_s)$, i.e., until a shock forms, if it does. Thus $u(x,t) \equiv 0$ in the space-time region $|x| \geq 1$.

However, when $|\tau| < 1$ we have $f(\tau) = 1 - \tau^2$, so that

$$\tau = x - t\left(1 - \tau^2\right) \Longleftrightarrow t\tau^2 - \tau + x - t = 0$$

$$\Longrightarrow \tau(x,t) = \frac{1 - \sqrt{1 + 4t(t-x)}}{2t} = \frac{2(x-t)}{1 + \sqrt{1 + 4t(t-x)}}$$

$$\Longrightarrow u(x,t) = 1 - \left(\frac{2(x-t)}{1 + \sqrt{1 + 4t(t-x)}}\right)^2. \qquad (1)$$

Observe that $\tau(x,0) = x$ and $u(x,0) = 1 - x^2$ (verifying that the correct initial condition is recovered for $|x| < 1$). The space-time region corresponding to $-1 < \tau < 1$ can be determined from

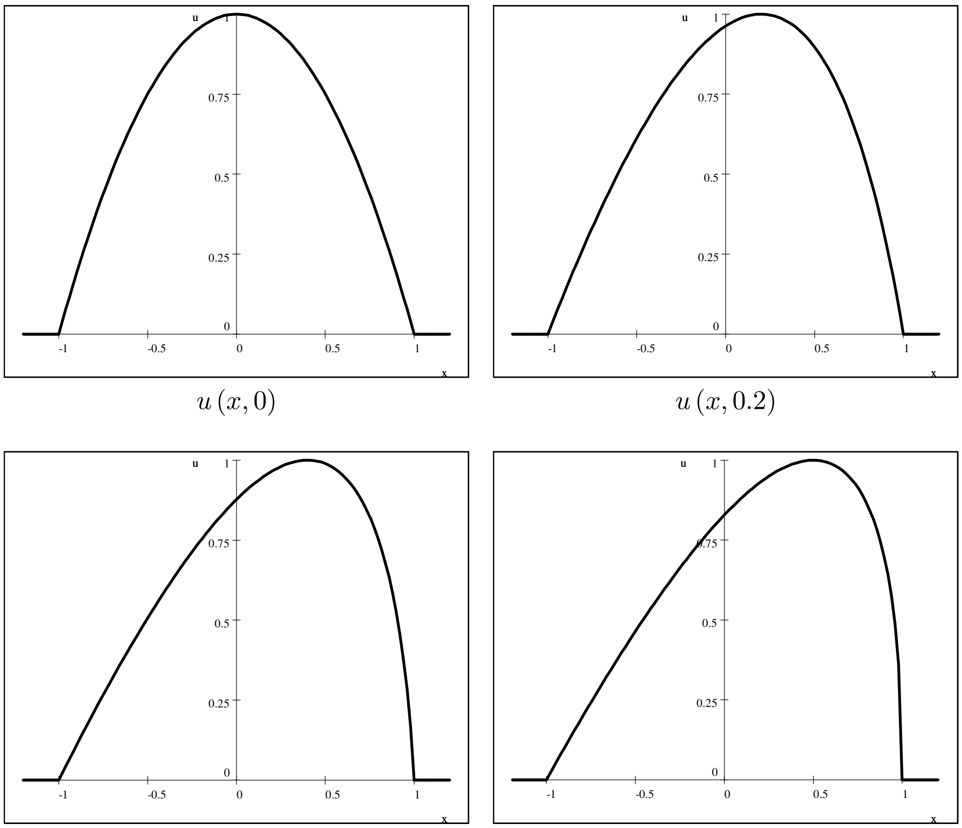
$$-1 < \frac{1 - \sqrt{1 + 4t(t-x)}}{2t} < 1 \Longleftrightarrow 1 - 2t < \sqrt{1 + 4t(t-x)} < 1 + 2t.$$

If $t \leq \frac{1}{2}$, this implies

$$1 - 4t + 4t^2 = (1-2t)^2 < 1 + 4t(t-x) = 1 + 4t^2 - 4xt < (1+2t)^2 = 1 + 4t + 4t^2$$

$$\Longrightarrow -4t < -4xt < 4t \Longleftrightarrow -1 < x < 1.$$

As we will show in a moment, a shock first forms at $(x,t) = \left(1, \frac{1}{2}\right)$. Thus, there is not a continuous solution for $t \geq \frac{1}{2}$. However, a *generalized or weak solution* does exist to the pde for $t \geq \frac{1}{2}$, but the derivation of this solution is beyond the scope of this course.



$u\,(x,0)$ $u\,(x,0.2)$ $u\,(x,0.4)$ $u\,(x,0.5)$

Fig. 2. Sequence of snapshots of $u\,(x,t)$ for selected values of t for $-1.2 \leq x \leq 1.2$.

A shock forms the first time $|u_x| \to \infty$. Clearly, this can never happen in the region $|x| > 1$ since $u \equiv 0$ there. It follows from the solution in the region $-1 < x < 1$, given by

$$u\,(x,t) = f\,(\tau) \text{ where } \tau = x - tf\,(\tau)\,,$$

that

$$u_x\,(x,t) = f'\,(\tau)\,\tau_x = \frac{f'\,(\tau)}{1 + tf'\,(\tau)} \Longrightarrow t_s = \min_{\tau} \frac{-1}{f'\,(\tau)} = \min_{\tau} \frac{1}{2\tau}$$

$$\Longrightarrow \tau_{\min} = +1 \Longrightarrow t_s = \tfrac{1}{2},\ x_s = \tau_{\min} + t_s f(\tau_{\min}) = 1 \Longrightarrow u(x_s, t_s) = 0.$$

A shock therefore first forms at the point $(x_s, t_s) = \left(1, \frac{1}{2}\right)$ and has zero amplitude.

Figure 2 shows a sequence of six "snapshots" of the solution for $u(x,t)$ for $t = 0$, 0.2, 0.4 and 0.5, respectively, for $-1.2 \leq x \leq 1.2$. As we go from $t = 0$ to $t = 0.5$ we see the rightward propagation and steepening in $|x| < 1$. The $t = 0.5$ panel shows the solution at the moment of shock formation where $|u_x| \to \infty$ at $x = 1$.

Question 2: The pde is

$$u_{xx} + 4u_{xy} + 3u_{yy} + 2(u_x + u_y) = -4\exp(x - y)$$

$$u(x, 0) = 3\left(x^2 + \sin 3x\right)e^x,$$

$$u_x(x, 0) + 3u_y(x, 0) = -6x(x + 1)e^x.$$

The characteristic variables are the level curves associated with

$$\frac{dy}{dx} = -\omega \text{ where } \omega^2 + 4\omega + 3 = 0 \Longrightarrow \omega = \frac{-4 \pm \sqrt{16 - 12}}{2} = -3 \text{ and } -1.$$

$$\Longrightarrow \left(\frac{dy}{dx}\right)_\xi = 1 \Longrightarrow \xi = x - y, \text{ and } \left(\frac{dy}{dx}\right)_\eta = 3 \Longrightarrow \eta = 3x - y,$$

with the inverse relations

$$x = \frac{\eta - \xi}{2} \text{ and } y = \frac{\eta - 3\xi}{2}.$$

The derivatives map according to

$$u_x = u_\xi + 3u_\eta,\ u_y = -(u_\xi + u_\eta),\ u_{xy} = -(u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}),$$

$$u_{xx} = u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta},\ u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Hence introducing the transformation $u = u(\xi, \eta)$ into the pde and initial conditions leads to the *H1* canonical form

$$u_{\xi\eta} - u_\eta = e^\xi, \tag{2}$$

$$u(\xi, 3\xi) = 3\left(\xi^2 + \sin 3\xi\right)e^\xi,\ u_\xi(\xi, 3\xi) = 3\xi(\xi + 1)e^\xi, \tag{3}$$

where we have appreciated that $y = 0 \Longrightarrow \eta = 3\xi$ and $x = \xi$.

It follows from (2) that

$$u_\xi - u = \left(\eta + \widehat{F}(\xi)\right)e^\xi \Longrightarrow \frac{\partial}{\partial \xi}\left(e^{-\xi}u\right) = \eta + \widehat{F}(\xi)$$

$$\Longrightarrow u(\xi, \eta) = (\xi\eta + F(\xi) + H(\eta))e^\xi, \tag{4}$$

where $\widehat{F}(\xi)$ and $F(\xi)$ are arbitrary functions of ξ and $H(\eta)$ is an arbitrary function of η. The general solution to the pde in terms of ξ and η is given by (4). Application of the initial data (3a) leads to,

$$u(\xi, 3\xi) = \left(3\xi^2 + F(\xi) + H(3\xi)\right) e^{\xi} = 3\left(\xi^2 + \sin 3\xi\right) e^{\xi}$$

$$\Longrightarrow F(\xi) + H(3\xi) = 3\sin(3\xi),$$

and from (3b) leads to

$$u_{\xi}(\xi, 3\xi) = u(\xi, 3\xi) + (\xi + F'(\xi)) e^{\xi}$$

$$= 3\left(\xi^2 + \sin 3\xi\right) e^{\xi} + (3\xi + F'(\xi)) e^{\xi} = 3\xi(\xi + 1) e^{\xi}$$

$$\Longrightarrow F'(\xi) = -3\sin(3\xi) \Longrightarrow F(\xi) = \cos(3\xi),$$

so that $H(3\xi)$ is given by

$$H(3\xi) = 3\sin(3\xi) - \cos(3\xi) \Longrightarrow H(\eta) = 3\sin(\eta) - \cos(\eta),$$

and thus

$$u(\xi, \eta) = (\xi\eta + \cos(3\xi) + 3\sin(\eta) - \cos(\eta)) e^{\xi},$$

and substituting in the relations $\xi = x - y$ and $\eta = 3x - y$, we get

$$u = ((x - y)(3x - y) + \cos(3(x - y)) + 3\sin(3x - y) - \cos(3x - y)) \exp(x - y).$$

Question 3: The pde is

$$\mathbf{u}_y + A\mathbf{u}_x = -\mathbf{u}, \text{ where } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}. \tag{5}$$

To show that this 3×3 system is totally hyperbolic we compute

$$\det(I - \omega A) = 0 \Longleftrightarrow \det\left(A - \tfrac{1}{\omega} I\right) = 0 \Longleftrightarrow \det \begin{bmatrix} 1 - \frac{1}{\omega} & 0 & 1 \\ 0 & 2 - \frac{1}{\omega} & 3 \\ 0 & 0 & -1 - \frac{1}{\omega} \end{bmatrix} = 0$$

$$\Longleftrightarrow \left(1 - \tfrac{1}{\omega}\right)\left(2 - \tfrac{1}{\omega}\right)\left(1 + \tfrac{1}{\omega}\right) = 0 \Longrightarrow \omega \in \left\{-1, \tfrac{1}{2}, 1\right\}.$$

Since the ω are real and distinct, the system is totally hyperbolic.

The characteristic curves are the level curves associated with

$$\frac{dy}{dx} = \omega,$$

i.e.,

$$\left(\frac{dy}{dx}\right)_{\xi_1} = 1 \Longrightarrow \xi_1 = y - x,$$

$$\left(\frac{dy}{dx}\right)_{\xi_2} = \frac{1}{2} \Longrightarrow \xi_2 = 2y - x,$$

$$\left(\frac{dy}{dx}\right)_{\xi_3} = -1 \Longrightarrow \xi_3 = y + x.$$

To reduce the system to canonical form we need to find the right eigenvectors of A associated with each eigenvalue ω^{-1}. Proceeding systematically, we have

$$\text{for } \omega = 1,\ \left(A - \tfrac{1}{\omega}I\right)\mathbf{r}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \mathbf{0} \Longrightarrow \mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\text{for } \omega = \frac{1}{2},\ \left(A - \tfrac{1}{\omega}I\right)\mathbf{r}_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -3 \end{bmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \mathbf{0} \Longrightarrow \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\text{for } \omega = -1,\ \left(A - \tfrac{1}{\omega}I\right)\mathbf{r}_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \mathbf{0} \Longrightarrow \mathbf{r}_3 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

Introduce the matrix R given by

$$R = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \Longrightarrow R^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

Introduce the new dependent variables $\mathbf{u} \equiv R\mathbf{v}$. Substitution into (5) leads to

$$R\mathbf{v}_y + AR\mathbf{v}_y = -R\mathbf{v} \Longleftrightarrow \mathbf{v}_y + R^{-1}AR\mathbf{v}_y = -\mathbf{v} \Longleftrightarrow \mathbf{v}_y + D\mathbf{v}_y = -\mathbf{v},$$

where

$$D = R^{-1}AR = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

If the initial data is given by

$$\mathbf{u}(x, x) = \mathbf{f}(x) = \left(\phi(x),\, 1,\, -e^{x}\right)^{\top}, \tag{6}$$

with $\phi(x)$ satisfying $\phi(0) = 0$, it follows that the initial data curve is the characteristic $\xi_1 = 0$. Thus a solution will only exist if a certain compatibility holds, which will determine $\phi(x)$. And we note that if a solution does exist, then there are infinitely many solutions. It follows from (6) that

$$\mathbf{u}_x(x, x) + \mathbf{u}_y(x, x) = \mathbf{f}' = \left(\phi'(x),\, 0,\, -e^{x}\right)^{\top}$$

$$\Longrightarrow A\mathbf{u}_x(x, x) + A\mathbf{u}_y(x, x) = A\mathbf{f}',$$

which if (5), evaluated on $y = x$, is used to eliminate $A\mathbf{u}_x(x,x)$, leads to

$$(A - I)\,\mathbf{u}_y(x,x) = \mathbf{f} + A\mathbf{f}'. \tag{7}$$

Let $\mathbf{l} = \begin{pmatrix} l_1 & l_2 & l_3 \end{pmatrix}$ be a left eigenvector of A associated with the eigenvalue $\omega^{-1} = 1$, i.e.,

$$\mathbf{l}\cdot(A - I) = \begin{pmatrix} l_1 & l_2 & l_3 \end{pmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} = \mathbf{0} \Longrightarrow \mathbf{l} = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix}.$$

Hence it follows from (7) that

$$\mathbf{l}\cdot(A - I)\cdot\mathbf{u}_y(x,x) = 0 = \mathbf{l}\cdot(\mathbf{f} + A\mathbf{f}') = \mathbf{l}\cdot(\mathbf{f} + \mathbf{f}') = 2\left(\phi' + \phi\right) - 2e^x$$

$$\Longrightarrow \phi' + \phi = e^x \Longrightarrow \frac{d}{dx}\left(e^x\phi\right) = e^{2x} \Longrightarrow e^x\phi = \int_0^x e^{2\beta}\,d\beta$$

$$\Longrightarrow \phi(x) = \frac{e^x - e^{-x}}{2} = \sinh(x).$$

Question 4: The pde is

$$u_t - u_{xx} - u_x + \beta u = 0, \text{ where } \beta \in \mathbb{R}.$$

Assume that

$$u = a\exp(ikx - \lambda t) + c.c.,$$

where the wavenumber $k \in \mathbb{R}$, the "growth rate" λ and amplitude a are both possibly complex-valued. Substitution into the pde implies

$$\lambda = -k^2 - \beta + ik \Longrightarrow \operatorname{Re}(\lambda) = -k^2 - \beta \Longrightarrow \Omega = \max_k\left(-k^2 - \beta\right) = -\beta.$$

We note that $\beta < 0 \Longrightarrow \Omega > 0$, so unstable. And $\beta > 0 \Longrightarrow \Omega < 0$, so asymptotically stable. And $\beta = 0 \Longrightarrow \Omega = 0$, so neutrally stable. The pde is well-posed in the sense of Hadamard for every real value of β.