Mathematics 436 Final Examination Lecturer: Professor G. E. Swaters

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Instructions. Please answer all 4 questions. Each question is worth 25 points.

1. Suppose $\{\varphi_k(x)\}_{k=1}^{\infty}$ is an orthonormal sequence of square-integrable functions defined on $x \in G \subset \mathbb{R}^n$ with the inner product

$$(u, w) = \int_G \rho u w \, dx,$$

with $\rho = \rho(x) > 0$.

   (a) If $\varphi(x)$ is a square-integrable function for $x \in G$, define the *Fourier Series* for $\varphi(x)$ with respect to $\{\varphi_k(x)\}_{k=1}^{\infty}$.

   (b) Beginning with the n^{th} partial sum associated with the Fourier Series for $\varphi(x)$, denoted by $\psi_n(x)$, show that *Bessel's Inequality* holds, i.e.,

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq \|\varphi\|^2.$$

   (c) Define what it means for a sequence of square-integrable functions $\{\psi_n(x)\}_{n=1}^{\infty}$ to *converge* to a function $\varphi(x)$ *in the mean.*

   (d) Show that *convergence in the mean* is equivalent to *Parseval's Identity.*

2. Let $\mathcal{L}$ be a *positive, self-adjoint, real-valued partial differential operator* defined for smooth square-integrable functions $f(x)$ where $x \in G \subset \mathbb{R}^n$ and satisfying the boundary conditions

$$\alpha f + \beta \frac{\partial f}{\partial n} = 0, \text{ with } \alpha, \beta \geq 0 \text{ where } \alpha + \beta > 0, \text{ for } x \in \partial G.$$

   (a) Define what it means for $\mathcal{L}$ to be a *positive, self-adjoint operator* with respect to the inner product (f, g).

   (b) Show that the solution, assuming it exists, to

$$u_{tt} + \mathcal{L}u = F(x, t), \; x \in G, \; t > 0,$$

$$u(x, 0) = f(x), \; u_t(x, 0) = g(x) \text{ for } x \in G,$$

$$\text{and } \alpha u + \beta \frac{\partial u}{\partial n} = B(x, t) \text{ for } x \in \partial G, \; t > 0,$$

   is unique.

3. Consider the wave equation in spherical coordinates written in the form

$$u_{tt} - c^2 \left[u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} \left(u_{\phi\phi} + \cot(\phi) u_\phi + \csc^2(\phi) u_{\theta\theta} \right) \right] = 0,$$

where $0 \le r < 1$, $0 \le \phi \le \pi$, $0 \le \theta < 2\pi$ and $t > 0$.

(a) Assuming $u = v(r,t)\cos(\phi)$, show that

$$v_{tt} - c^2 \left(v_{rr} + \frac{2}{r} v_r - \frac{2}{r^2} v \right) = 0. \tag{1}$$

The radial eigenfunctions associated with (1) are the solutions to the *spherical Bessel equation of order one*, which in self-adjoint form, is given by

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr} \right) + \left(\lambda^2 r^2 - 2 \right) R = 0. \tag{2}$$

The solution to (2) that is bounded at $r = 0$ is the *spherical Bessel function of the first kind of order one,* given by

$$R(r) = j_1(\lambda r).$$

(b) Show that

$$R(1) = j_1(\lambda) = 0 \Longleftrightarrow \tan(\lambda) = \lambda.$$

Let the countable infinity of positive solutions to this relation be denoted by $\{\lambda_n\}_{n=1}^{\infty}$ where $0 < \lambda_1 < \lambda_2 \ \cdots$.

(c) Show that

$$(j_1(\lambda_n r), j_1(\lambda_m r)) = \int_0^1 j_1(\lambda_n r)\, j_1(\lambda_m r)\, r^2\, dr = 0 \text{ if } n \neq m.$$

(d) Show that

$$\int_0^1 j_1^2(\lambda_n r)\, r^2\, dr = \tfrac{1}{2} j_0^2(\lambda_n).$$

4. Use an eigenfunction expansion and the results of Question 3 to solve

$$v_{tt} - c^2 \left(v_{rr} + \frac{2}{r} v_r - \frac{2}{r^2} v \right) = (1 - r)\sin(t), \ \ 0 \le r < 1, \ \ t > 0,$$

$$v(r,0) = v_t(r,0) = v(1,t) = 0.$$

Hint: You may assume that $j_1(1/c) \neq 0$. Thus, the solution is given by

$$v(r,t) = \frac{4}{c} \sum_{n=1}^{\infty} \frac{\left[\sin(c\lambda_n t) - c\lambda_n \sin(t) \right] j_1(\lambda_n r)}{\lambda_n^2 \left(1 - c^2 \lambda_n^2 \right) \left[1 + \cos(\lambda_n) \right]}, \text{ where } j_1(\lambda_n) = 0.$$

Useful Formulae Sheet

Let $j_n(x)$ be the *spherical Bessel function of the first kind of order* n, then

$$(2n+1)\frac{d}{dx}j_n(x) = nj_{n-1}(x) - (n+1)j_{n+1}(x),$$

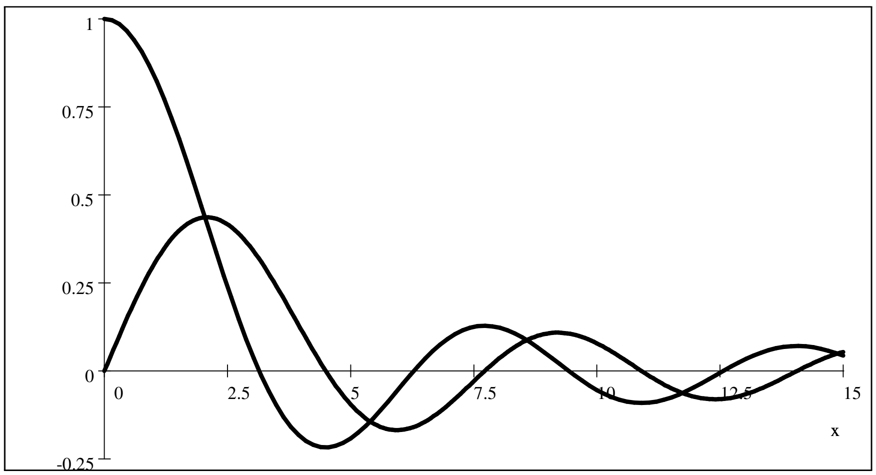
$$j_{n+1}(x) + j_{n-1}(x) = \frac{2n+1}{x}j_n(x),$$

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+1/2}(x).$$

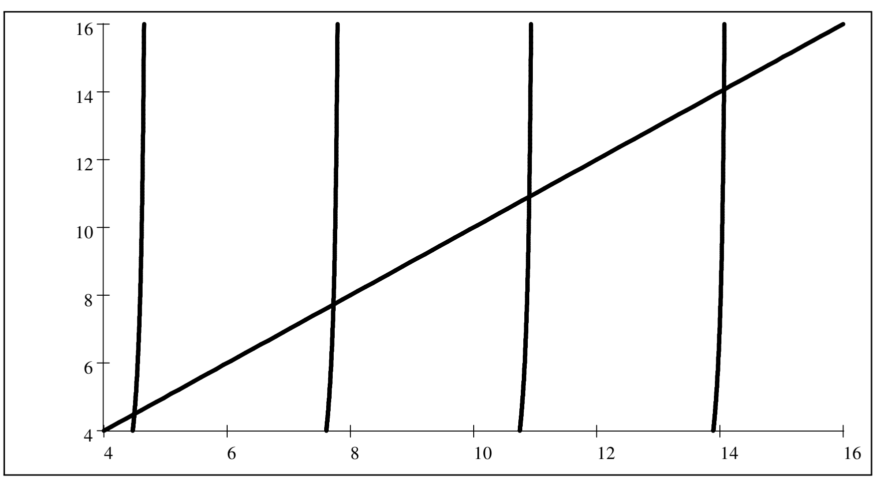
In particular,

$$j_0(x) = \frac{\sin(x)}{x} \text{ and } j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x},$$

$$j_1(x) = -\frac{d}{dx}j_0(x), \quad \lim_{x\to 0} j_0(x) = 1 \text{ and } \lim_{x\to 0} j_1(x) = 0,$$



Plot of $j_0(x)$ and $j_1(x)$ *vs.* x.



Plot of $\tan(\lambda)$ and λ *vs.* λ. The positive intersection points between these curves are the solutions of $\tan(\lambda) = \lambda$. The first four solutions are given by $\lambda_1 \simeq 4.48$, $\lambda_2 \simeq 7.73$, $\lambda_3 \simeq 10.9$ and $\lambda_4 \simeq 14.07$, respectively.