Solutions for Math 436 2006 Final

Question 1a: The Fourier Series is defined as

$$\varphi(x) = \sum_{k=1}^{\infty} (\varphi, \varphi_k)\ \varphi_k(x).$$

Question 1b: The n^{th} partial sum is given by

$$\psi_n(x) = \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x).$$

To show Bessel's Inequality, we begin with

$$0 \leq \|\varphi(x) - \psi_n(x)\|^2 = (\varphi - \psi_n, \varphi - \psi_n) = (\varphi, \varphi) - 2(\varphi, \psi_n) + (\psi_n, \psi_n)$$

$$= (\varphi, \varphi) - 2\left(\varphi, \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right) + \left(\sum_{m=1}^{n} (\varphi, \varphi_m)\ \varphi_m(x), \sum_{k=1}^{n} (\varphi, \varphi_k)\ \varphi_k(x)\right)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_k) + \sum_{m=1}^{n}\sum_{k=1}^{n} (\varphi, \varphi_k)(\varphi, \varphi_m)(\varphi_m, \varphi_k)$$

$$= (\varphi, \varphi) - 2\sum_{k=1}^{n} (\varphi, \varphi_k)^2 + \sum_{k=1}^{n} (\varphi, \varphi_k)^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2$$

$$\Longrightarrow \sum_{k=1}^{n} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).$$

Since the right-hand-side of this expression is independent of n, this inequality must hold for all n regardless of large it is, and thus in the limit

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 \leq (\varphi, \varphi).$$

Question 1c: Mean square convergence is defined as

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0.$$

Question 1d: We must show that

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\| = 0 \Longleftrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi).$$

From Question 1b, we have

$$\|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \sum_{k=1}^{n} (\varphi, \varphi_k)^2.$$

Thus, provided the limit exists,

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = (\varphi, \varphi) - \lim_{n\to\infty} \sum_{k=1}^{n} (\varphi, \varphi_k)^2 .$$

Hence

$$\lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0 \Longrightarrow \sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi),$$

and

$$\sum_{k=1}^{\infty} (\varphi, \varphi_k)^2 = (\varphi, \varphi) \Longrightarrow \lim_{n\to\infty} \|\varphi(x) - \psi_n(x)\|^2 = 0.$$

Question 2a: $\mathcal{L}$ is a positive self-adjoint operator if $\forall\ f(x)$ and $g(x)$ that satisfy the boundary conditions,

$$(f, \mathcal{L}f) \geq 0 \text{ and } (f, \mathcal{L}g) = (g, \mathcal{L}f),$$

respectively.
Question 2b: Assume two solutions exist to the problem denoted by $u_1(x,t)$ and $u_2(x,t)$, respectively, i.e.,

$$\partial_{tt} u_1 + \mathcal{L}u_1 = F(x,t),\ x \in G,\ t > 0,$$

$$u_1(x,0) = f(x),\ \partial_t u_1(x,0) = g(x) \text{ for } x \in G,$$

$$\text{and } \alpha u_1 + \beta \frac{\partial u_1}{\partial n} = B(x,t) \text{ for } x \in \partial G,\ t > 0,$$

and

$$\partial_{tt} u_2 + \mathcal{L}u_2 = F(x,t),\ x \in G,\ t > 0,$$

$$u_2(x,0) = f(x),\ \partial_t u_2(x,0) = g(x) \text{ for } x \in G,$$

$$\text{and } \alpha u_2 + \beta \frac{\partial u_2}{\partial n} = B(x,t) \text{ for } x \in \partial G,\ t > 0.$$

Define the difference $w = u_1 - u_2$, it follows that w satisfies

$$w_{tt} + \mathcal{L}w = 0,\ x \in G,\ t > 0,$$

$$w(x,0) = 0,\ w_t(x,0) = 0 \text{ for } x \in G,$$

$$\text{and } \alpha w + \beta \frac{\partial w}{\partial n} = 0 \text{ for } x \in \partial G,\ t > 0.$$

We form the energy equation

$$w_t w_{tt} + w_t \mathcal{L}w = 0 \Longrightarrow \partial_t \|w_t\|^2 + 2(w_t, \mathcal{L}w) = 0,$$

and since $\mathcal{L}$ is a *self-adjoint operator* it follows that $(w_t, \mathcal{L}w) = (w, \mathcal{L}w_t)$, so that the energy equation can be written in the form

$$\partial_t \|w_t\|^2 + (w_t, \mathcal{L}w) + (w, \mathcal{L}w_t) = 0 \Longrightarrow \frac{\partial}{\partial t}\left[\|w_t\|^2 + (w, \mathcal{L}w)\right] = 0$$

$$\Longrightarrow \|w_t\|^2 + (w, \mathcal{L}w) = \left[\|w_t\|^2 + (w, \mathcal{L}w)\right]_{t=0} = 0$$

and since $\mathcal{L}$ is a *positive operator* it follows that $(w, \mathcal{L}w) \geq 0$ so that this equation implies

$$\|w_t\| = 0 \Longrightarrow w_t = 0 \Longrightarrow w(x,t) = w(x,0) = 0.$$

Question 3a: The result follows from direct substitution upon noting that

$$u_{\phi\phi} + \cot(\phi)\, u_\phi = -2\cos(\phi)\, v(r,t).$$

Question 3b: From the Useful Formulae sheet, we have

$$j_1(\lambda) = 0 = \frac{\sin(\lambda)}{\lambda^2} - \frac{\cos(\lambda)}{\lambda} \Longrightarrow \tan(\lambda) = \lambda.$$

Question 3c: To show the orthogonality relationship we begin with the pair of equations

$$\frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_n r)}{dr}\right) + \left(\lambda_n^2 r^2 - 2\right) j_1(\lambda_n r) = 0, \tag{1}$$

$$\frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_m r)}{dr}\right) + \left(\lambda_m^2 r^2 - 2\right) j_1(\lambda_m r) = 0. \tag{2}$$

Multiplying (1) by $j_1(\lambda_m r)$ and (2) $j_1(\lambda_n r)$ and subtracting, we get

$$\left(\lambda_m^2 - \lambda_n^2\right) r^2 j_1(\lambda_n r)\, j_1(\lambda_m r)$$

$$= j_1(\lambda_m r) \frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_n r)}{dr}\right) - j_1(\lambda_n r) \frac{d}{dr}\left(r^2 \frac{dj_1(\lambda_m r)}{dr}\right),$$

which if we integrate with respect to r over the interval $(0,1)$, yields

$$\left(\lambda_m^2 - \lambda_n^2\right) \int_0^1 r^2 j_1(\lambda_n r)\, j_1(\lambda_m r)\, dr$$

$$= \left[r^2 \left(j_1(\lambda_m r) \frac{dj_1(\lambda_n r)}{dr} - j_1(\lambda_n r) \frac{dj_1(\lambda_m r)}{dr}\right)\right]_0^1 = 0$$

$$\Longrightarrow \int_0^1 r^2 j_1(\lambda_n r)\, j_1(\lambda_m r)\, dr = 0 \text{ if } \lambda_m \neq \lambda_n, \text{ i.e., } n \neq m.$$

Question 3d: To show this relationship, we use the formula

$$j_1(x) = -\frac{d}{dx} j_0(x).$$

So that

$$\int_0^1 r^2 j_1^2(\lambda_n r)\, dr = -\frac{1}{\lambda_n} \int_0^1 r^2 j_1(\lambda_n r) \frac{d}{dr} j_0(\lambda_n r)\, dr$$

$$= -\frac{1}{\lambda_n}\left[r^2 j_1(\lambda_n r) j_0(\lambda_n r)\right]_0^1 + \frac{1}{\lambda_n}\int_0^1 j_0(\lambda_n r)\frac{d}{dr}\left[r^2 j_1(\lambda_n r)\right] dr$$

$$= \frac{1}{\lambda_n^3}\int_0^1 \frac{\sin(\lambda_n r)}{r}\frac{d}{dr}\left[\frac{\sin(\lambda_n r)}{\lambda_n} - r\cos(\lambda_n r)\right] dr$$

$$= \frac{1}{\lambda_n^3}\int_0^1 \frac{\sin(\lambda_n r)}{r}\left[\cos(\lambda_n r) - \cos(\lambda_n r) + r\lambda_n \sin(\lambda_n r)\right] dr$$

$$= \frac{1}{\lambda_n^2}\int_0^1 \sin^2(\lambda_n r)\ dr = \frac{1}{2\lambda_n^2}\int_0^1 1 - \cos(2\lambda_n r)\ dr$$

$$= \frac{2\lambda_n - \sin(2\lambda_n)}{4\lambda_n^3} = \frac{\lambda_n - \sin(\lambda_n)\cos(\lambda_n)}{2\lambda_n^3} = \frac{\sin^2(\lambda_n)}{2\lambda_n^2} = \tfrac{1}{2} j_0^2(\lambda_n).$$

Question 4: The pde is

$$v_{tt} - c^2\left(v_{rr} + \frac{2}{r}v_r - \frac{2}{r^2}v\right) = \sin(t)(1-r),\ \ 0 \le r < 1,\ \ t > 0, \tag{2}$$

$$v(r,0) = v_t(r,0) = v(1,t) = 0.$$

Observing that the boundary conditions are homogeneous Dirichlet conditions, it follows from Question 3 that we may construct a solution of the form

$$v(r,t) = \sum_{n=1}^{\infty} A_n(t) j_1(\lambda_n r),\ \text{where } j_1(\lambda_n) = 0. \tag{3}$$

Substitution of (3) into (2) leads to

$$\sum_{n=1}^{\infty}\left[A_n'' + (c\lambda_n)^2 A_n\right] j_1(\lambda_n r) = \sin(t)(1-r)$$

$$\Longrightarrow \left[A_n'' + (c\lambda_n)^2 A_n\right]\int_0^1 r^2 j_1^2(\lambda_n r)\ dr = \sin(t)\int_0^1 r^2(1-r) j_1(\lambda_n r)\ dr$$

for $n = 1, 2, \cdots$. Thus, using the result from Question 3d, we have for each A_n

$$A_n'' + (c\lambda_n)^2 A_n = -\frac{2\sin(t)}{\lambda_n j_0^2(\lambda_n)}\int_0^1 \left(r^2 - r^3\right)\frac{d}{dr} j_0(\lambda_n r)\ dr$$

$$= \frac{2\sin(t)}{\lambda_n j_0^2(\lambda_n)}\int_0^1 \left(2r - 3r^2\right) j_0(\lambda_n r)\ dr = \frac{2\sin(t)}{\lambda_n^2 j_0^2(\lambda_n)}\int_0^1 (2-3r)\sin(\lambda_n r)\ dr$$

$$= -\frac{2\sin(t)}{\lambda_n^3 j_0^2(\lambda_n)}\int_0^1 (2-3r)\frac{d}{dr}\cos(\lambda_n r)\ dr$$

$$= \frac{2\sin(t)}{\lambda_n^3 j_0^2(\lambda_n)}\left\{\left[(3r-2)\cos(\lambda_n r)\right]_0^1 - 3\int_0^1 \cos(\lambda_n r)\ dr\right\}$$

$$= \frac{2\sin(t)}{\lambda_n^3 j_0^2(\lambda_n)} \left\{ \cos(\lambda_n) + 2 - \frac{3\sin(\lambda_n)}{\lambda_n} \right\} = \frac{4\sin(t)\left[1-\cos(\lambda_n)\right]}{\lambda_n^3 j_0^2(\lambda_n)}$$

$$= \frac{4\sin^2(\lambda_n)\sin(t)}{\lambda_n^3 \left(\sin(\lambda_n)/\lambda_n\right)^2 \left[1-\cos(\lambda_n)\right]} = \frac{4\sin(t)}{\lambda_n\left[1+\cos(\lambda_n)\right]}.$$

Hence, we must solve

$$A_n'' + (c\lambda_n)^2 A_n = \frac{4\sin(t)}{\lambda_n\left[1+\cos(\lambda_n)\right]} \text{ subject to } A_n(0) = A_n'(0) = 0.$$

Assuming that there is no n for which $c\lambda_n = 1$ (the is equivalent to assuming that $j_0(1/c) \neq 0$), the solution for A_n that satisfies $A_n(0) = 0$ is of the form

$$A_n(t) = \alpha \sin(c\lambda_n t) - \frac{4\sin(t)}{\lambda_n\left(1-c^2\lambda_n^2\right)\left[1+\cos(\lambda_n)\right]},$$

where α is a free constant. Application of $A_n'(0) = 0$ leads to

$$\alpha c\lambda_n - \frac{4}{\lambda_n\left(1-c^2\lambda_n^2\right)\left[1+\cos(\lambda_n)\right]} = 0$$

$$\Longrightarrow \alpha = \frac{4}{c\lambda_n^2\left(1-c^2\lambda_n^2\right)\left[1+\cos(\lambda_n)\right]}.$$

$$\Longrightarrow A_n(t) = \frac{4\left[\sin(c\lambda_n t) - c\lambda_n \sin(t)\right]}{c\lambda_n^2\left(1-c^2\lambda_n^2\right)\left[1+\cos(\lambda_n)\right]}.$$

Hence, the solution can be written in the form

$$v(r,t) = \frac{4}{c}\sum_{n=1}^{\infty} \frac{\left[\sin(c\lambda_n t) - c\lambda_n \sin(t)\right] j_1(\lambda_n r)}{\lambda_n^2\left(1-c^2\lambda_n^2\right)\left[1+\cos(\lambda_n)\right]}, \text{ where } j_1(\lambda_n) = 0.$$