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A note on the stability theory of buoyancy-driven ocean currents over a sloping bottom

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1. Introduction

Ocean fronts correspond to geophysical fluid flows for which there are sharp horizontal gradients in the temperature and velocity fields (see Figure 1). Examples of these types of flows include upwelling and surface coupled fronts typically seen over continental shelves and buoyancy-driven currents produced by large river out flows. These currents possess regions with relatively large horizontal gradients. This kinematic property has made the development of a stability theory for these flows somewhat difficult since one cannot, in principle, apply classical quasigeostrophic instability theory (e.g., Pedlosky, 1987; see also Paldor and Killworth, 1987). Recently, Swaters (1993) developed and analyzed an intermediate-lengthscale model for the baroclinic evolution of these flows. This theory filtered out the Rayleigh-like barotropic instabilities and focussed on the inherently baroclinic destabilization of buoyancy-driven ocean fronts, while respecting the essential kinematic properties of these currents.

Swaters derived two sets of general stability results for this new model. The first set of stability conditions was derived in the context of normal-mode perturbations and the second set of general Liapunov stability conditions was derived via an appropriately constrained energy invariant. However, the two sets of stability conditions were not completely isomorphic to each other and this was a problematic aspect of the analysis from the author's point of view.

The principal purpose of this note is to show that the normal-mode stability conditions derived by Swaters can be obtained by working with an appropriately constrained linear momentum invariant and that these stability results can be generalized to establish conditions for the nonlinear stability in the sense of Liapunov for buoyancy-driven ocean fronts. In addition, as a technical side issue, by working with the linear momentum invariant, we show that it is possible to dispense with the undesirable *Poincaré* inequality that Swaters was forced to introduce using an energy invariant. This mathematical result is of particular physical interest in applying the theory to, for example, buoyancy-driven flows along a coastline for which no Poincaré inequality exists since the domain may be considered as effectively unbounded.

The plan of this note is as follows. In §2, we briefly introduce the Swaters (1993) model. The stability analysis is most properly understood as a consequence of the underlying noncanonical Hamiltonian structure of the governing equations. This structure is introduced in §2, as are the associated invariants needed in our stability theory.

In §3, we establish a variational principle for steady buoyancy-driven fronts based on an appropriately constrained linear momentum invariant. We show that the normal-mode stability results presented by Swaters (1993) are sufficient to ensure that the second variation of the constrained momentum invariant evaluated at the steady solution is definite. Assuming these hypotheses, it is straightforward to establish the linear stability in the sense of Liapunov of these flows. Finally, in §3, we generalize these linear stability conditions to state sufficient convexity hypotheses on the constrained linear momentum invariant which will establish the nonlinear stability in the sense of Liapunov of these flows.

In §4, we compare the stability results presented here with those of Swaters (1993). We also interpret our results in the context of the potential vorticity gradients associated with the steady flow and make some concluding remarks.

2. Problem formulation and Hamiltonian structure

2.1. Governing equations

Since a detailed derivation of the equation has already been given (see Swaters, 1993) we will be brief in our presentation. The basic model is a two layer shallow water system with both layers inviscid and incompressible. Assuming that the aspect ratio $h*/H$, where $h*$ is a thickness scale for the frontal layer and H is a thickness scale for the lower layer (see Figure 1), is a smaller parameter, i.e.,

$$0 < \frac{h*}{H} \ll 1,$$

it is possible to show that the leading order dynamics are described by

$$\begin{aligned}&(\Delta p + h)_t + \partial(p, \Delta p + h - sy) = 0,\\ &h_t + \partial\left(p + h\ \Delta h + \frac{1}{2}\nabla h \cdot \nabla h, h\right) = 0,\end{aligned} \tag{2.1}$$

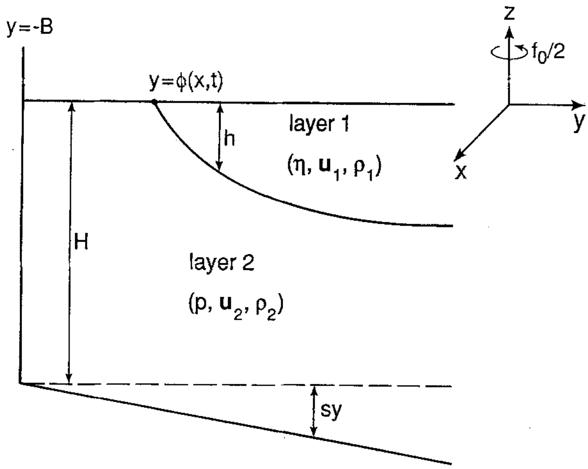

Figure 1
The geometry of the two-layer model used in this paper.

with the associated geostrophic velocities

$$\begin{aligned} \boldsymbol{u}_1 &= \hat{e}_3 \times \nabla h, \\ \boldsymbol{u}_2 &= \hat{e}_3 \times \nabla p, \end{aligned} \tag{2.2}$$

where p, h, s $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the lower layer pressure, frontal height, bottom slope, upper layer velocity and lower layer velocity, respectively, and where $\partial(A, B) \equiv A_x B_y - A_y B_x$. The inclusion of the upper layer advective terms results in the nonlinear frontal height terms seen in the second equation of (2.1). A similar model appropriate for a midlatitude β-plane has been derived independently by Cushman-Roisin et al. (1992).

The spatial domain associated with $p(x, y, t)$ will be given by $R = \{x_L < x < x_R,\ -B < y\}$, which qualitatively represents a domain with a coast at $y = -B$ and periodic in x (see Figure 1). The domain of the front associated with $h(x, y, t) > 0$ is given by $F = \{x_L < x < x_R,\ -B < \phi_1(x, t) < y < \phi_2(x, t) \leq \infty\}$ where the functions $\phi_1(x, t)$ and $\phi_2(x, t)$ mark the boundary or outcroppings of the front.

The boundary conditions are given by

$$\begin{aligned} &p = \lambda \qquad \text{on } y = -B, \\ &\|\nabla p\| \to 0 \qquad \text{as } y \to \infty, \\ &h = 0 \qquad \text{on } y = \phi_{1,2}, \end{aligned} \tag{2.3}$$

where the condition at $y = \phi_2$ is replaced by

$$\|\nabla h\| \to 0 \qquad \text{as } y \to \infty,$$

when $\phi_2 = \infty$, with the smooth periodicity conditions

$$(p, h, \phi_{1,2})|_{x=x_L} = (p, h, \phi_{1,2})|_{x=x_R}.$$

These boundary conditions specify no normal flow on the coast, zero velocity far from the coast, and frontal height of zero at an outcropping, respectively. It should be noted that the boundaries $y = \phi_{1,2}$ are dynamic and so conditions also apply to their evolution. These do not come into play in our analysis (see Swaters, 1993).

2.2. *Hamiltonian formulation*

The derivation of our stability is most concisely presented as a consequence of the underlying Hamiltonian structure of the governing equations. The model (2.1) can be cast into the noncanonical Hamiltonian form (see Swaters, 1993, Benjamin, 1984, or Olver, 1982)

$$\boldsymbol{q}_t = J \frac{\delta H}{\delta \boldsymbol{q}}, \tag{2.4}$$

where $\delta H/\delta \boldsymbol{q}$ is the variational derivative of the Hamiltonian, H, with respect to $\boldsymbol{q}$,

$$H(\boldsymbol{q}) \equiv \frac{1}{2} \iint_R \nabla p \cdot \nabla p \, dx \, dy - \frac{1}{2} \iint_F h \, \nabla h \cdot \nabla h \, dx \, dy - \lambda \int_{\partial R} \nabla p \cdot \boldsymbol{n} \, dS, \tag{2.5}$$

$J = [J_{ij}]$ is a 2×2 matrix of differential operators whose components are given by

$$J_{ij} = -\delta_{i1}\delta_{j1}\partial(q_1 - sy, *) + \delta_{i2}\delta_{j2}\partial(q_2, *), \tag{2.6}$$

where δ_{mm} is the Kronecker delta function and $\boldsymbol{q} = (q_1, q_2)^T$ with

$$q_1 \equiv \Delta p + h, \qquad q_2 \equiv h. \tag{2.7}$$

It is straightforward to check that $H(\boldsymbol{q})$ is an invariant of the dynamics (see Swaters, 1993). Note that the second term in (2.5) is cubically nonlinear. It is the presence of this term that necessitated the introduction of a Poincaré inequality in the energy-based stability argument of Swaters (1993).

Alternatively, we may write the system (2.1) in the Poisson bracket notation

$$\boldsymbol{q}_t = [\boldsymbol{q}, H], \tag{2.8}$$

where

$$[F, G] \equiv \left\langle \frac{\delta F}{\delta \boldsymbol{q}}, J \frac{\delta G}{\delta \boldsymbol{q}} \right\rangle, \tag{2.9}$$

is the Poisson bracket for arbitrary functionals F and G and where $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is the inner product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \int_R \boldsymbol{a}\boldsymbol{b}^T \, dx \, dy.$$

2.3. *Casimir and momentum invariants*

The *Casimirs* are those conserved quantities that lie in the kernel of the Poisson bracket, that is, they satisfy

$$[F, C] = 0,$$

for all sufficiently smooth functionals $F(\boldsymbol{q})$. Using the definition of the bracket (2.9) it follows that

$$J \frac{\delta C}{\delta \boldsymbol{q}} \equiv \boldsymbol{0},$$

giving the general solution

$$C(\boldsymbol{q}) = \iint_R \Phi_1(q_1 - sy) \, dx \, dy + \iint_F \Phi_2(q_2) \, dx \, dy, \tag{2.10}$$

where Φ_1 and Φ_2 are sufficiently smooth functions of their arguments.

We can find other invariants of the system using Noether's Theorem (see Courant and Hilbert, 1962). For our purposes, all we need is the invariant associated with the fact that H and J are invariant under translations in x. This invariant, denoted M, satisfies (see Benjamin, 1984)

$$J \frac{\delta M}{\delta \boldsymbol{q}} \equiv -\boldsymbol{q}_x, \tag{2.11}$$

giving the general solution (modulo a Casimir)

$$M(\boldsymbol{q}) = \iint_R y(q_1 - q_2) \, dx \, dy = \iint_R y \, \Delta p \, dx \, dy. \tag{2.12}$$

The functional M corresponds to the x component of linear momentum for the lower layer, as can be seen by writing M as

$$\begin{aligned} M(\boldsymbol{q}) &= \iint_R y(v_{2_x} - u_{2_y}) \, dx \, dy \\ &= \iint_R u_2 \, dx \, dy + B \int_{x_L}^{x_R} u_2(x, -B, t) \, dx, \end{aligned}$$

where we have integrated by parts and exploited the boundary conditions. This invariant is also referred to as the x component of "Kelvin's impulse" (Benjamin, 1984). Note that M as given by (2.12) does not contain the cubic nonlinear frontal height terms seen in (2.5). It is this property which will obviate the need for a Poincaré inequality in the following analysis.

3. Stability of parallel shear flow solutions

The model (2.1) with (2.3) admits the parallel shear flow solutions $h = h_0(y)$ and $p = p_0(y)$. In this section we shall give sufficient conditions for the linear and nonlinear stability of such flows. The stability will be determined in the sense of Liapunov, that is we will determine conditions which provide an *a priori* bound on the perturbation norm, $\|\delta \boldsymbol{q}\|$, in terms of a multiple of the same norm evaluated at $t = 0$, written as $\|\delta \boldsymbol{q}\|$.

To establish stability we first present a variational principle for the steady flows based on the constrained linear momentum invariant, also referred to as the pseudomomentum, given by

$$\mathscr{M} = M + C, \tag{3.1}$$

where M is the linear momentum invariant given by (2.12) and C is the Casimir given by (2.10). We choose the Casimir density functions so that the first order necessary condition for the steady solutions to be an extremal of $\mathscr{M}$, i.e., $\delta\mathscr{M}(p_0, h_0) = 0$, is satisfied. We note here that a similar variational principle has been used by Ripa (1983) to establish general stability conditions for zonal flows in a one-layer model on a β-plane or a sphere.

3.1. *Variational principle*

It follows from (3.1) that

$$\delta\mathscr{M} = \iint_R [(\Phi_1'(q_1 - sy) + y)\,\delta q_1 + (\Phi_2'(q_2) - y)\,\delta q_2]\,dx\,dy. \tag{3.2}$$

And thus, parallel shear flow solutions of the form $p_0(y)$ and $h_0(y)$ satisfy the first-order necessary condition, $\delta\mathscr{M}(p_0, h_0) = 0$, for extremizing the constrained linear momentum invariant, $\mathscr{M}$, provided the Casimir densities are chosen to satisfy

$$\begin{aligned} &\Phi_1'(p_{0_{yy}} + h_0 - sy) = -y, \\ &\Phi_2'(h_0) = y. \end{aligned} \tag{3.3}$$

That is, the Casimir density functions are chosen so that

$$p_{0_{yy}} + h_0 - sy = \tilde{\Phi}_1(y), \tag{3.4}$$

$$h_0 = \tilde{\Phi}_2(y), \tag{3.5}$$

where $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are the inverse functions associated with Φ_1' and Φ_2', respectively. We will discuss the restrictions associated with the existence of the inverse functions $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ in §4.

3.2. Linear stability

The variational principle can be exploited to derive linear stability conditions as follows. The second variation of $\mathscr{M}$ is given by

$$\delta^2\mathscr{M}(p_0, h_0) = \iint_R \{[\Phi_1''(p_{0_{yy}} + h_0 - sy)](\Delta\delta p + \delta h)^2 + [\Phi_2''(h_0)](\delta h)^2\}\, dx\, dy. \tag{3.6}$$

It is straightforward to verify that $\delta^2\mathscr{M}(p_0, h_0)$ is an invariant of the linear equations obtained by substituting $h(x, y, t) = h_0(y) + \delta h(x, y, t)$ and $p(x, y, t) = p_0(y) + \delta p(x, y, t)$ into (2.1) and neglecting quadratic terms in the perturbation quantities.

Since $\delta^2\mathscr{M}(p_0, h_0)$ is an invariant of the linear stability problem, it follows that linear stability can be established if conditions can be found on Φ_1'' and Φ_2'' so that $\delta^2\mathscr{M}(p_0, h_0)$ is definite for all perturbations δp and δh. It follows from (3.6) that $p_0(y)$ and $h_0(y)$ are linearly stable in the sense of Liapunov with respect to the perturbation norm

$$\|\delta \boldsymbol{q}\|^2 = \iint_R (\Delta\delta p + \delta h)^2 + (\delta h)^2\, dx\, dy, \tag{3.7}$$

if the Casimir density functions Φ_1 and Φ_2 as determined by (3.3) satisfy either

$$\Phi_1''(p_{0_{yy}} + h_0 - sy) > 0, \quad \text{and} \quad \Phi_2''(h_0) > 0, \tag{3.8}$$

or

$$\Phi_1''(p_{0_{yy}} + h_0 - sy) < 0, \quad \text{and} \quad \Phi_2''(h_0) < 0. \tag{3.9}$$

for all $y \in [-B, \infty)$ or $y \in [\phi_1, \phi_2]$ as appropriate.

Clearly, conditions (3.8) guarantee that $\delta^2\mathscr{M}(p_0, h_0)$ is positive definite while conditions (3.9) guarantee that $\delta^2\mathscr{M}(p_0, h_0)$ is negative definite (which establishes the formal stability of the solution; see Holm et al., 1985). In order to establish linear stability, all that remains is to show if conditions (3.8) or (3.9) hold, it is possible to *a priori* bound the perturbation norm.

Assuming (3.9) holds, we have from (3.6) and (3.7) that

$$\Gamma_1 \|\delta \boldsymbol{q}\|^2 \leq \delta^2\mathscr{M}(p_0, h_0) \leq \Gamma_2 \|\delta \boldsymbol{q}\|^2, \tag{3.10}$$

where

$$\Gamma_1 = \min(\inf_R \Phi_{10}'', \inf_F \Phi_{20}'') > 0 \quad \text{and} \quad \Gamma_2 = \max(\sup_R \Phi_{10}'', \sup_F \Phi_{20}'') > 0.$$

It then follows from (3.10), and the invariance of $\delta^2\mathcal{M}(p_0, h_0)$ that

$$\|\delta \boldsymbol{q}\| \le (\Gamma_2/\Gamma_1)^{1/2}\|\delta\tilde{\boldsymbol{q}}\|,$$

where $\|\delta\tilde{\boldsymbol{q}}\| \equiv \|\delta\boldsymbol{q}\|_{t=0}$.

Similarly if (3.9) holds we have $\|\delta\boldsymbol{q}\| \le K\|\delta\tilde{\boldsymbol{q}}\|$, where

$$K^2 = \min(\inf_R \Phi_{10}'', \inf_F \Phi_{20}'')/[\max(\sup_R \Phi_{10}'', \sup_F \Phi_{20}'')] > 0.$$

These *a priori* estimates establish linear stability provided (3.8) or (3.9) hold.

3.3. Nonlinear stability

To establish the nonlinear stability of these parallel shear flow solutions, we must consider the total variation of the pseudomomentum, $\Delta\mathcal{M}$, given by

$$\begin{aligned}\Delta\mathcal{M} = {} & M(p_0+\delta p, h_0+\delta h) - M(p_0, h_0)\\ & + C(p_0+\delta p, h_0+\delta h) - C(p_0, h_0),\end{aligned} \tag{3.11}$$

where M and C are given by (2.12) and (2.10) respectively, with the Casimir densities determined by (3.3). We note that $\Delta\mathcal{M}$ is conserved by the full non-linear dynamics (2.1). Equation (3.11) can be expressed in the form

$$\begin{aligned}\Delta\mathcal{M} = {} & \iint_R \{\Phi_1(q_{10}+\delta q_1) - \Phi_1(q_{10}) - \Phi_1'(q_{10})\,\delta q_1\}\,dx\,dy\\ & + \iint_F \{\Phi_2(h_0+\delta h) - \Phi_2(h_0) - \Phi_2'(h_0)\,\delta h\}\,dx\,dy,\end{aligned} \tag{3.12}$$

where we introduced $q_{10} \equiv p_{0_{yy}} + h_0 - sy$ for notational convenience.

Suppose that the convexity conditions

$$\begin{aligned}&\alpha_1 \le \Phi_1''(\xi) \le \beta_1,\\ &\alpha_2 \le \Phi_2''(\xi) \le \beta_2,\end{aligned} \tag{3.13}$$

hold for all arguments ξ, where α_1, β_1, α_2, and β_2 are real numbers. Conditions (3.13) can be integrated twice and substituted into (3.12) to give

$$\iint_R \{\alpha_1(\delta q_1)^2 + \alpha_2(\delta h)^2\}\,dx\,dy \le 2\,\Delta\mathcal{M} \le \iint_R \{\beta_1(\delta q_1)^2 + \beta_2(\delta h)^2\}\,dx\,dy. \tag{3.14}$$

It follows that if the Casimir densities $\Phi_1(\xi)$ and $\Phi_2(\xi)$, which are determined by the parallel shear flow solutions $p_0(y)$ and $h_0(y)$ through the relations (3.3), satisfy either

$$0 < \alpha_1 \le \Phi_1''(\xi) \le \beta_1 < \infty, \quad \text{and} \quad 0 < \alpha_2 \le \Phi_2''(\xi) \le \beta_2 < \infty, \tag{3.15}$$

or

$$-\infty < \alpha_1 \leq \Phi_1''(\xi) \leq \beta_1 < 0, \quad \text{and} \quad -\infty < \alpha_2 \leq \Phi_2''(\xi) \leq \beta_2 < 0, \tag{3.16}$$

for all ξ where α_1, β_1, α_2, and β_2 are some real constants, then the parallel shear flow solutions $p_0(y)$ and $h_0(y)$ are nonlinearly stable in the sense of Liapunov with respect to the disturbance norm $\|\delta \boldsymbol{q}\|$ given by (3.7).

Clearly conditions (3.15) and equation (3.14) establish that

$$0 < \min(\alpha_1, \alpha_2)\|\delta \boldsymbol{q}\|^2 \leq 2\,\Delta\mathcal{M} \leq \max(\beta_1, \beta_2)\|\delta \boldsymbol{q}\|^2 < \infty$$

and therefore that

$$\|\delta \boldsymbol{q}\| \leq \left[\frac{\max(\beta_1, \beta_2)}{\min(\alpha_1, \alpha_2)}\right]^{1/2} \|\delta \tilde{\boldsymbol{q}}\|$$

giving nonlinearly stability in the sense of Liapunov. Similarly, when conditions (3.16) hold it follows from (3.14) that

$$0 < -\max(\beta_1, \beta_2)\|\delta \boldsymbol{q}\|^2 \leq -2\Delta\mathcal{M}(\boldsymbol{q}) \leq -\min(\alpha_1, \alpha_2)\|\delta \boldsymbol{q}\|^2 < \infty$$

and therefore that

$$\|\delta \boldsymbol{q}\| \leq \left[\frac{\min(\alpha_1, \alpha_2)}{\max(\beta_1, \beta_2)}\right]^{1/2} \|\delta \tilde{\boldsymbol{q}}\|$$

giving nonlinearly stability in the sense of Liapunov.

4. Discussion and conclusions

From a physical point of view, it is important to interpret the linear stability results in the context of the mean potential vorticity gradients. It follows from (3.3) that

$$\Phi_1''(p_{0_{yy}} + h_0 - sy) = \frac{1}{U_{0_{yy}} - h_{0_y} + s},$$

$$\Phi_2''(h_0) = \frac{1}{h_{0_y}},$$

where $U_0(y) \equiv -dp_0/dy$ is the x-direction velocity in layer one. Thus conditions (3.8) are equivalent to

$$U_{0_{yy}} - h_{0_y} + s > 0, \quad \forall y \in [-B, \infty) \quad \text{and} \quad h_{0_y} > 0, \quad \forall y \in [\phi_1, \phi_2] \tag{4.1}$$

and conditions (3.9) are equivalent to

$$U_{0_{yy}} - h_{0_y} + s < 0, \quad \forall y \in [-B, \infty) \quad \text{and} \quad h_{0_y} < 0, \quad \forall y \in [\phi_1, \phi_2]. \tag{4.2}$$

Physically, these conditions establish the stability of parallel shear flows if the transverse vorticity gradients in each layer are everywhere of the same sign and are *exactly* the conditions presented in the normal-mode linear stability analysis of Swaters (1993). However, deriving the stability conditions (4.1) and (4.2) as a consequence of the underlying Hamiltonian structure has the distinct advantage that it is clear how to generalize (4.1) and (4.2) to establish conditions for the *nonlinear* stability of the steady solutions. In addition, these conditions do *not* require the existence of a Poincaré inequality and thus can be applied to a coastal domain which is unbounded in the offshore direction.

The conditions (4.1) and (4.2) imply that $p_{0_{yy}} + h_0 - sy$ and h_0 are monotonic functions. Under this restriction, their inverse functions exists. Thus, the form of the Casimir densities given in (3.4) and (3.5) do not restrict the solutions that the linear theorem can be applied to. Since any flow which is nonlinearly stable is necessarily linearly stable, there are also no restrictions placed on the nonlinear results. While theoretically inverse functions can be found, in practice it may prove to be difficult.

The conditions (3.15) and (3.16) cannot be recast into a form involving the parallel shear flow variables since the conditions must hold for all arguments ξ. This also implies that the nonlinear conditions are much stricter than the linear conditions. This is best explained through an example. Consider the flow given by

$$h_0(y) = 1 - \frac{e^{-y}}{2},$$

$$p_0(y) = \frac{e^{-y}}{2},$$

where we have taken $\phi_1 = -1$, $\phi_2 = \infty$ and $s = 1$. The flow represents an isolated front at $y = -1$ with an exponentially decreasing velocity in the lower layer. It follows from (4.1) that this flow is linearly stable. From (3.4) and (3.5) we can solve for the Casimir densities to get

$$\Phi_1'(\xi) = \xi - 1 \quad \text{and} \quad \Phi_2'(\xi) = -\ln(1-\xi)$$

and thus obtain that

$$\Phi_1''(\xi) = 1 \quad \text{and} \quad \Phi_2''(\xi) = (1-\xi)^{-1}.$$

Since $\Phi_2''(\xi)$ can be both positive and negative, the flow cannot meet either of the nonlinear stability conditions (3.15) or (3.16). While the linear stability condition (3.8) is required to hold only for $0 \leq \xi < 1$, that is when ξ takes on the values of $h_0(y)$, the nonlinear condition must be satisfied for all positive ξ, since ξ now corresponds to the perturbed frontal height. Thus, having transverse potential vorticity gradients everywhere of the same sign in each layer does *not* guarantee nonlinear stability. It should be noted that flows which do not satisfy the stability conditions are not necessarily unstable. In order to better understand both the stability conditions and the characteristics of possible instabilities a numerical analysis is necessary.

In conclusion, we have shown that through the use of the invariant associated with the x component of linear momentum it is possible to do a complete study of the stability of parallel shear flows for the geostrophic model developed by Swaters (1993). By using the linear momentum functional, the stability analysis does not require the existence of a Poincaré inequality. As well, the linear stability conditions presented in this paper are identical to the normal mode results of Swaters (1993).

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Abstract

In a previous paper, a new model was derived describing the baroclinic dynamics of buoyancy-driven ocean currents over a sloping bottom. In particular, a normal mode stability analysis and a general stability analysis based on an appropriately constrained energy invariant were presented. However, these two sets of stability results were not identical to each other. Here, we show that the normal-mode stability results previously described may be derived from a general stability analysis based on an appropriately constrained linear momentum invariant. In addition, we establish conditions for the nonlinear stability in the sense of Liapunov of these flows. The analysis presented here eliminates the need to introduce a Poincaré inequality between the perturbation energy and the enstrophy which the previous analysis was forced to assume. Relaxing this assumption means the present analysis is applicable to a much larger range of flow geometries and is therefore a substantially stronger result.

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