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On stationary baroclinic planetary dipole-vortices in a continuously-stratified fluid of finite depth

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1. Introduction

Modons are isolated dipole-vortex solutions of the quasi-geostrophic equations of motion (e.g., Stern, 1975; Larichev and Reznik, 1976; Flierl et al., 1980). It has been suggested (McWilliams, 1980; Flierl et al., 1983; Butchart et al., 1989; Haines and Marshall, 1987; Haines, 1989) that β-plane modons may be useful prototype models for atmospheric blocking; particularly for those blocking configurations which resemble a dipole such as that frequently seen over the Northeast Atlantic ocean. Since the real atmosphere and ocean are obviously baroclinic, it is of considerable interest to construct modon-like solutions for the continuously-stratified quasi-geostrophic equations. However, Haines and Marshall (1987) and Butchart et al. (1989) have qualitatively established that the horizontal amplitude field associated with the gravest mode of a vertical normal mode decomposition of the blocking streamfunction, for a continuously-stratified fluid of finite depth, *necessarily* contains a downstream Rossby wave-tail if the background zonal flow is eastward. While explicitly pointing out this very important conclusion, the above authors did not determine the detailed structure of the streamfunction/vorticity fields. The principal purpose of the present paper is to present an analytical solution for a stationary modon embedded in an eastward baroclinic zonal flow in a continuously-stratified fluid depth on a β-plane which satisfies the appropriate radiation condition in the upstream flow region and to discuss its dynamical characteristics.

It is hoped that the solution presented here will be useful in analyzing the detailed structure of actual baroclinic blocks. The solution presented here should also be useful in plasma physics where modon solutions have been proposed as models for certain aspects of the convective motion associated with observed anomalous heat transports in fusion-containment devices. The method of solution presented here will also be relevant for modon perturbation and modulation theory in the situation where the modon is undergoing non-adiabatic adjustment and a generated field of external Rossby waves will be an intrinsic aspect of the time-dependent behavior.

The plan of the paper is as follows. In Section 2 the governing equations are given and the normal mode decomposition are described. For completeness, we briefly review the gravest-mode radiation theorem of Haines and Marshall (1987) and Butchart et al. (1989) using an alternate argument based on a Rayleigh-Ritz variational principle. In Section 3 we present our solution for the radiating baroclinic modon which satisfies the appropriate upstream radiation condition. In Section 4 we discuss some of the dynamical characteristics of the solution. The paper concludes with Section 5, which contains a summary and a few closing remarks.

2. Governing equations and normal mode decomposition

The *nondimensional* baroclinic quasi-geostrophic equations for a continuously-stratified incompressible Boussinesq fluid of finite depth can be written in the form

$$[\Delta p + (N^{-2}p_z)_z]_t + \beta p_x + J[p, \Delta p + (N^{-2}p_z)_z] = 0, \tag{2.1a}$$

with the *vertical* boundary conditions

$$p_{zt} + J(p, p_z) = 0, \qquad \text{on } z = 0, 1, \tag{2.1b, c}$$

where the notation is standard (e.g., Pedlosky, 1987). The geostrophic pressure is given by $p = p(x, y, z, t)$ with horizontal geostrophic velocity $\boldsymbol{v} \equiv (u, v) = \hat{e}_3 \times \nabla p \equiv (-p_y, p_x)$, where (x, y, z, t) are the eastward, northward, vertically upward, and time coordinates, respectively. The nondimensional beta-parameter is given by $\beta \equiv \beta^* U_*^2/L$ where (β^*, U_*, L) correspond to the dimensional scales for beta, horizontal velocity field and horizontal lengthscale, respectively. The nondimensional Burger's parameter (or equivalently the nondimensional squared Brunt-Vaisala frequency) is given by $N^2(z) \equiv (H/f_0 L)^2(-g\varrho_*^{-1}\, d\bar{\varrho}/dz) > 0$ (stable stratification) where g, ϱ_*, $\bar{\varrho}(z)$, H and f_0 are the gravitational constant, constant reference density, variable hydrostatic background density, mean depth of the fluid, and constant Coriolis parameter, respectively. The Jacobian is given by $J(A, B) \equiv A_x B_y - B_x A_y$ and subscripts with respect to (x, y, z, t) indicate the appropriate partial derivative. In addition, $\Delta \equiv \partial_{xx}^2 + \partial_{yy}^2$. The vertical lengthscale is the depth of the fluid so that $z = 0, 1$ corresponds to the (flat) bottom and (rigid) top or surface, respectively, of the fluid. The boundary conditions (2.1b, c) express the no-normal flow constraint required on the surface and bottom of the fluid, respectively.

We seek a *stationary* solution to (2.1) of the form

$$p = -yU(z) + \varphi(x, y, z), \tag{2.2}$$

where $\varphi(x, y, z)$ will be called the eddy streamfunction. The term $-yU(z)$ corresponds to a baroclinic eastward ($U(z) \geq 0$) mean zonal current that is not sheared with respect to y. We are interested in an *isolated* solution. By isolated we mean, following Flierl et al. (1980), a solution for φ satisfying $|\varphi| \to 0$ as $x^2 + y^2 \to \infty$ for which there exist regions containing closed *total* streamlines of p. The fluid within these regions cannot exit and is thus isolated from the exterior region fluid.

Substitution of (2.2) into (2.1a) leads to

$$J[-yU + \varphi, \Delta\varphi + (N^{-2}\varphi_z)_z + \{\beta - (N^{-2}U_z)_z\}y] = 0, \tag{2.3a}$$

which can be immediately integrated to imply

$$-yU + \varphi = \mathscr{F}[\Delta\varphi + (N^{-2}\varphi_z)_z + \{\beta - (N^{-2}U_z)_z\}y, z], \tag{2.3b}$$

where the exact form of the function $\mathscr{F}(*, z)$ is yet-to-be-determined. Because the Jacobian operator involves only explicit (x, y)-differentiation, it follows that the function $\mathscr{F}(*, z)$ can have an explicit dependence on the vertical coordinate z.

2.1. *Form of $\mathscr{F}(*, z)$ in the exterior region*

For all those streamlines which extend to infinity, (2.3b) implies

$$-yU(z) = \mathscr{F}[\{\beta - (N^{-2}U_z)_z\}]y, z],$$

or, equivalently

$$\mathscr{F}(*, z) = \{U/[(N^{-2}U_z)_z - \beta]\}*, \tag{2.4}$$

where we have explicitly used the fact that $|\varphi| \to 0$ as $x^1 + y^2 \to \infty$ for all $z \in (0, 1)$. As in classical barotropic modon theory, we consider the situation where all the streamlines in the region $r \equiv (x^2 + y^2)^{1/2} > a$ extend to infinity and call this the *exterior* region. The cylinder $r = a$ is called the modon boundary or radius. It is convenient to represent $\mathscr{F}$ given by (2.4) in terms of the mean streamfunction $p_0 \equiv -U(z)y$ and potential vorticity $q_0 \equiv \Delta p_0 + (N^{-2}p_{0_z})_z + \beta y = \{\beta - (N^{-2}U_z)_z]y$ in the form

$$\mathscr{F}(*, z) = */\Lambda_0(z), \quad \text{for } r > a, \tag{2.5a}$$

where

$$\Lambda_0 \equiv \partial q_0/\partial p_0 = [(N^{-2}U_z)_z - \beta]/U. \tag{2.5b}$$

We may interpret (2.5b) as defining $\Lambda_0(z)$, or equivalently defining $U(z)$ given a potential vorticity/streamfunction relationship

$$\Lambda_0 = \partial q_0 / \partial p_0. \tag{2.6}$$

Because of the profound relationships between Λ_0 and the Hamiltonian structure of the governing equations (e.g., Holm, 1986), Arnol'd's stability criteria (Swaters, 1986; McIntyre and Shepherd, 1987), and potential vorticity/streamfunction scatter diagrams as a diagnostic tool in free-mode theory (Read et al., 1986), it is more convenient to adopt the latter view. Thus, given $\Lambda_0(z)$, we determine $U(z)$ such that

$$(N^{-2}U_z)_z - \Lambda_0 U = \beta, \tag{2.7}$$

subject to appropriate boundary conditions which will be specified later.

Substitution of (2.5) into (2.3b) implies that for all those streamlines which extend to infinity

$$\Delta\varphi + (N^{-2}\varphi_z)_z - \Lambda_0\varphi = 0. \tag{2.8}$$

It is important to emphasize that even though (2.8) is completely linear in the eddy streamfunction, any solution to (2.8) when substituted into (2.2) yields a *fully nonlinear* solution to (2.1).

2.2. Form of $\mathscr{F}(*, z)$ in the interior region

For those streamlines which do not extend to infinity, corresponding to the region $r < a$, it is not possible to use the far field structure to determine the form of $\mathscr{F}(*, z)$. Again, as in classical barotropic modon theory, we call this region the *interior* region. In the interior region we assume

$$\mathscr{F}(*, z) = [\Lambda_0(z) - \mu(z)]^{-1}*, \qquad r < a, \tag{2.9a}$$

where the additional parameter $\mu(z)$ is, in general, a function of the vertical coordinate. Substitution of (2.9a) into (2.3b) implies that the *total* streamfunction satisfies

$$\Delta p + (N^{-2}p_z)_z + (\mu - \Lambda_0)p = -\beta y, \qquad r < a. \tag{2.9b}$$

Again, we emphasize, even though (2.9b) is linear, any p which solves (2.9b) is a fully nonlinear solution to (2.1). It is more convenient to work with the total streamfunction rather than the eddy streamfunction in the modon interior. The dimensional analogue of the parameter μ has the dimensions of inverse length squared. In classical barotropic modon theory, $\mu^{1/2}$ is often referred to as the modon wavenumber and we shall refer to $\mu^{1/2}$ similarly.

2.3. Vertical boundary conditions

If (2.2) is substituted into (2.1b, c), it follows that

$$-yU + \varphi = \mathscr{F}_1(-yU_z + \varphi_z), \tag{2.10a}$$

$$-yU + \varphi = \mathscr{F}_0(-yU_z + \varphi_z), \tag{2.10b}$$

on $z = 1$ and $z = 0$, respectively, where the functions $\mathscr{F}_{1,0}(*)$ are yet-to-be determined. For all those streamlines which extend to infinity (2.10a) and (2.10b) imply

$$\mathscr{F}_1(*) = \frac{U}{U_z} *, \tag{2.11a}$$

$$\mathscr{F}_0(*) = \frac{U}{U_z} *, \tag{2.11b}$$

on $z = 1$ and $z = 0$, respectively, where we have again explicitly used the fact that $|\varphi| \to \infty$ as $r \to \infty$ on $z = 0, 1$. Substitution of (2.11) into (2.10) yields the exterior vertical boundary conditions in the form

$$U\varphi_z - U_z\varphi = 0, \qquad \text{on } z = 0, 1. \tag{2.12}$$

Here, again, it is not possible to use the far field flow structure to determine the form of $\mathscr{F}_1(*)$ and $\mathscr{F}_2(*)$ in the interior region $r < a$. Our choice is to impose (2.12) on the *entire* flow region. This ansatz has been made in oceanographic isolated eddy calculations (e.g., Hogg, 1980; Swaters and Mysak, 1985) and is implicit in the modelling work of Butchart et al. (1989). Adopting (2.12) as the uniform surface and bottom condition is consistent with requiring the continuity of pressure and normal mass flux on the surface and bottom, respectively, across the modon boundary $r = a$. The boundary condition (2.12) is written in terms of the eddy streamfunction $\varphi(x, y, z)$. While this is appropriate for the *exterior* region, in the *interior* region it will be more convenient to re-cast (2.12) into the form $Up_z - U_zp = 0$ on $z = 0, 1$.

2.4. Matching conditions on the modon boundary

The modon boundary forms a streamline, which will separate the interior and exterior regions. On $r = a$, we impose the conditions

$$(\varphi - Uy)_{r=a^+} = 0, \tag{2.13}$$

$$p|_{r=a^-} = 0. \tag{2.14}$$

These conditions will imply that the leading order geostrophic pressure and normal mass flux are continuous at the modon boundary. The implications of (2.13) and (2.14) on the continuity of the vorticity and azimuthal mass flux at the modon boundary will be discussed in Section 4.

2.5. Normal mode decomposition

If a separation of variable solution for the exterior eddy streamfunction of the form $\varphi(x, y, z) = \Pi(z)h(x, y)$ is introduced into (2.8) and (2.12), it follows that

$$\varphi = \sum_{n=0}^{\infty} \Pi_n(z)h_n(x, y), \tag{2.15}$$

where the orthonormalized vertical modes are determined from

$$(N^{-2}\Pi_n')' + (\lambda_n - \Lambda_0)\Pi_n = 0, \tag{2.16a}$$

$$U\Pi_n' - U'\Pi_n = 0, \qquad \text{on } z = 0, 1. \tag{2.16b}$$

$$\int_0^1 \Pi_n(z)\Pi_m(z)\, dz = \delta_{n,m}, \tag{2.16c}$$

for $n = 0, 1, 2, 3, \ldots$, where $(\ )' \equiv d(\ )/dz$ and $\delta_{n,m}$ is the Knonecker delta function between n and m. The eigenvalues λ_n form a discrete set $\{\gamma_n\}_{n=0}^{\infty}$ satisfying $-\infty < \lambda_0 < \lambda_1 < \cdots < \infty$. The corresponding horizontal problems are given by

$$\Delta h_n - \lambda_n h_n = 0, \qquad \text{for } r > a. \tag{2.17a}$$

The boundary condition (2.13) can be written in the form

$$\sum_{n=0}^{\infty} \Pi_n(z)h_n|_{r=a^+} = U(z)a \sin\theta, \tag{2.17b}$$

where $\tan\theta = y/x$.

If the minimum eigenvalue $\lambda_0 > 0$, then the solutions for h_n for $n \geq 0$ *all* exponentially decay as $r \to \infty$ and there are no Rossby waves in the exterior region. However, Butchart et al. (1989) have shown that, at least, $\lambda_0 < 0$ if $U(z) > 0$ and thus there are exterior Rossby waves associated with the gravest mode. For completeness, we give an alternate demonstration of this theorem via a Rayleigh-Ritz variational principle for the minimum eigenvalue of the problem (2.16). This result is closely related to earlier proofs of the non-existence of baroclinic inertial boundary currents in oceanic flows (e.g., Robinson, 1965; Pedlosky, 1965).

If we introduce the transformation $\Pi_n(z) = U(z)\Phi_n(z)$ into (2.16), it follows that

$$(U^2\Phi_n'/N^2)' + (\lambda_n U^2 + \beta U)\Phi_n = 0, \tag{2.18a}$$

$$\Phi_n' = 0 \qquad \text{on } z = 0, 1, \tag{2.18b}$$

$$\langle \Phi_n, \Phi_m \rangle \equiv \int_0^1 U^2\Phi_n\Phi_m\, dz = \delta_{n,m}. \tag{2.18c}$$

If (2.18a) is multiplied through by $\Phi_n(z)$ and the result integrated over $z \in (0, 1)$, it follows that

$$\lambda_n = -\beta \int_0^1 U\Phi_n^1\, dz + \int_0^1 U^2(\Phi_n')^2/N^2\, dz. \tag{2.19}$$

Thus, a Rayleigh-Ritz variational principle for λ_0 can be expressed in the form

$$\lambda_0 = \min_{\phi} \left\{ -\beta \int_0^1 U(z)\, dz + \int_0^1 U^2(\phi')^2/N^2\, dz \right\}, \tag{2.20}$$

where the minimization is carried out over all smooth functions $\phi(z)$ satisfying $\phi' = 0$ on $z = 0, 1$ and $\langle \phi, \phi \rangle = 1$. Clearly, one candidate is the constant $\phi = [\int_0^1 U^2(z)\, dz]^{-1/2}$, so that we can bound λ_0 by

$$\lambda_0 \le -\beta \int_0^1 U(z)\, dz \Big/ \int_0^1 U^2(z)\, dz. \tag{2.21}$$

From which we conclude that $\lambda_0 < 0$ if $U(z) > 0$ for all $z \in (0, 1)$.

This result is extremely important physically since it necessarily implies that it is not possible to obtain isolated eddy solutions to (2.1) which do not possess a Rossby wave field in the exterior region if the ambient mean flow is eastward (or westerly in the jargon of atmospheric dynamics). This is the flow geometry associated with mid-latitude blocking configurations. Haines and Marshall (1987) have pointed out, however, that this conclusion does not necessarily imply that continuously-stratified modon models cannot provide useful prototype blocking solutions since the decay timescale of the radially outward energy flux associated with the Rossby waves can still be long in comparison to the timescale associated with transient synoptic eddies. Thus the structure of the flow pattern in the blocking region can appear coherent for timescales comparable to observed blocks which is about 10 days (Rex, 1950). Indeed, it can be argued that the intrinsic energy flux associated with the exterior Rossby wave field provides a mechanism for the eventual decay of blocks more or less back into the averaged climatology.

The Rossby wave field in the exterior region must, however, satisfy the appropriate radiation condition in the *upstream* region which removes any incoming Rossby waves (Miles, 1968; McCartney, 1975) given by

$$\operatorname*{limit}_{r \to \infty} r^{1/2}\varphi = 0, \qquad \text{for all } \theta \in (\pi/2, 3\pi/2), \tag{2.22}$$

where we are assuming $U(z) > 0$.

In summary, the exterior eddy streamfunction $\varphi(x, y, z)$ will be obtained by solving (2.16), (2.17) together with the no-upstream waves constraint (2.22). The far-field mean zonal flow $U(z)$ is determined by (2.7) subject to the boundary conditions

$$U(0) = U_0 \ge 0, \tag{2.23a}$$

$$U(1) = U_1 \ge 0. \tag{2.23b}$$

In the *interior* ($r < a$) region, we obtain a solution for the *total* streamfunction $p(x, y, z)$ in the form

$$p = \sum_{n=0}^{\infty} \Pi_n(z) g_n(x, y). \tag{2.24}$$

Substitution of (2.24) into (2.9b) and (2.14) leads to, after exploiting (2.16), the sequence of horizontal problems given by

$$(\Delta - \lambda_n) g_n + \sum_{m=0}^{\infty} \left\{ \int_0^1 \mu(z) \Pi_n(z) \Pi_m(z)\, dz \right\} g_m = \gamma_n y, \tag{2.25a}$$

$$g_n \big|_{r=a} = 0, \tag{2.25b}$$

$$\gamma_n \equiv -\beta \int_0^1 \Pi_n(z)\, dz, \tag{2.25c}$$

for $n = 0, 1, 2, 3, \ldots$. Equation (2.25a) corresponds to an infinite set of *coupled* partial differential equations. We have not been able to obtain a general solution to (2.25a) for an arbitrary $\mu(z)$. However, further analytical progress can be made if we assume μ is a constant. If we make this approximation, the integral in (2.25a) is given by $\mu\delta_{nm}$ and the $g_n(x, y)$ functions will be determined by the *uncoupled* system of equations

$$(\Delta + \mu - \lambda_n) = \gamma_n y, \tag{2.26}$$

for $n = 0, 1, 2, 3, \ldots$. For the remainder of this paper it will be assumed that μ is constant so that (2.26) can be used to determine $g_n(x, y)$.

3. The solution and determination of the modon wavenumber

The solution for the nth-component of the horizontal part of the interior *total* streamfunction as determined by (2.26) subject to (2.25b, c) is straightforward to obtain and is given by

$$g_n(x, y) = \{ \nu_n \mathscr{L}_n(r) + \gamma_n (\mu - \lambda_n)^{-1} r \} \sin\theta, \tag{3.1a}$$

where

$$\nu_n \equiv a\gamma_n [(\lambda_n - \mu) \mathscr{L}_n(a)]^{-1}, \tag{3.1b}$$

$$\mathscr{L}_n(r) \equiv \begin{cases} J_1[(\mu - \lambda_n)^{1/2} r], & \text{if } \mu > \lambda_n, \\ I_1[(\lambda_n - \mu)^{1/2} r], & \text{if } \mu < \lambda_n. \end{cases} \tag{3.1c}$$

The solution for the nth-component of the horizontal part of the exterior *eddy* streamfunction as determined by (2.16), (2.17) and (2.22) can be written in the form

$$h_n(x, y) = \mathscr{K}_n \sum_{\ell=1}^{\infty} \alpha_\ell \{Y_\ell[(-\lambda_n)^{1/2}r] \sin(\ell\theta) + \chi_\ell(r, \theta)\}/Y_\ell[(-\lambda_n)^{1/2}a], \tag{3.2a}$$

$$\chi_\ell(r, \theta) = \sum_{m=1}^{\infty} b_{\ell,m} J_m[(-\lambda_n)^{1/2}r] \sin(m\theta), \tag{3.2b}$$

for all those n for which $\lambda_n < 0$; say, $n = 0, 1, 2, \ldots, N$ (we guaranteed that, at least, $\lambda_0 < 0$), and

$$h_n(x, y) = \mathscr{K}_n K_1[(\lambda_n)^{1/2}r] \sin\theta / K_1[(\lambda_n)^{1/2}a], \tag{3.2c}$$

for all $n > N$ for which $\lambda_n > 0$. There will be no cosine terms in (3.2a, b) and no higher sine harmonics in (3.2c) because of the boundary condition (2.17b). The fact that there are no cosine terms in (3.2) will imply that the downstream Rossby wave-tail will be an odd function about $y = 0$. This is precisely what Haines and Marshall (1987) have observed in their numerical simulations. The coefficients $\mathscr{K}_n$ and α_ℓ will be determined by the boundary condition (2.17b) and the coefficients $b_{\ell,m}$ are chosen to satisfy the no upstream waves condition (2.22).

For those eigenmodes in which $\lambda_n > 0$, the horizontal structure functions, given by (3.2c), decay exponentially rapidly at infinity and so trivially satisfy the no-upstream waves constraint (2.22). Physically, for these vertical modes there is no exterior Rossby wave contribution. For those eigenmodes for which $\lambda_n < 0$, the horizontal structure functions decay like $O(r^{-1/2})$ and thus the $\chi_\ell(r, \theta)$ contribution is necessary to satisfy (2.22).

Recalling that the even (odd) $J_\ell(*)$ functions have the same asymptotic behaviour as the odd (even) $Y_\ell(*)$ functions for $(*) \to \infty$ (Abramowitz and Stegun, 1964), it follows from (2.22) that the coefficients $b_{\ell,m}$ must satisfy the constraints

$$\sin(2\ell\theta) = \sum_{m=0}^{\infty} (-1)^{\ell+m+1} b_{2\ell,m+1} \sin[(2m+1)\theta], \tag{3.3a}$$

$$\sin[(2\ell+1)\theta] = \sum_{m=1}^{\infty} (-1)^{\ell+m} b_{2\ell+1,2m} \sin(2m\theta), \tag{3.3b}$$

for $\ell = 0, 1, 2, 3, \ldots$ in the interval $\theta \in (\pi/2, 3\pi/2)$. Exploiting the fact that *both* the sets $\{\sin(2\ell\theta)\}_{\ell=0}^{\infty}$ and $\{\sin[(2\ell+1)\theta]\}_{\ell=0}^{\infty}$ form complete the orthogonal sets of (odd) basis functions in the interval $\theta \in (\pi/2, 3\pi/2)$ (Miles, 1968), it follows that

$$b_{\ell,m} = \begin{cases} (4/\pi)\ell(m^2 - \ell^2)^{-1}, & (\ell \text{ even}, m \text{ odd}), \\ (4/\pi)m(m^2 - \ell^2)^{-1}, & (\ell \text{ odd}, m \text{ even}), \\ 0, & (m - \ell \text{ even}). \end{cases} \tag{3.4}$$

These relations completely determine the $b_{\ell,m}$ coefficients.

To determine the α_ℓ coefficients we rewrite the sum in (3.2a) as

$$h_n(x, y) = \mathscr{K}_n \sum_{m=1}^{\infty} \left\{ \sum_{\ell=1}^{\infty} \alpha_\ell \Gamma_{\ell,m}(r) \right\} \sin(m\theta), \tag{3.5a}$$

where

$$\Gamma_{\ell,m}(r) = \{\delta_{\ell,m} Y_\ell[(-\lambda_n)^{1/2} r] + b_{\ell,m} J_m[(-\lambda_n)^{1/2} r]\} / Y_\ell[(-\lambda_n)^{1/2} a]. \tag{3.5b}$$

In order to satisfy the boundary condition (2.17b) we choose that α_ℓ in (3.5a) to satisfy

$$\sum_{\ell=1}^{\infty} \alpha_\ell \Gamma_{\ell,m}(a) = \delta_{m,1}, \tag{3.6}$$

for $m = 1, 2, 3, \ldots$, when $\delta_{m,1}$ is the Kronecker delta between m and 1. The condition (3.6) determines the α_ℓ coefficients. As it turns out, relatively few α_ℓ need to be computed to be able to give a very good approximation in the infinity sums in (3.5). If we recall that $Y_\ell[(-\lambda_n)^{1/2} a] \to -\infty$ and $J_m[(-\lambda_n)^{1/2} a] \to 0$ as ℓ and $m \to \infty$, respectively (Abramowitz and Stegun, 1965), then $\Gamma_{\ell,m}(a) \simeq \delta_{\ell,m}$ for sufficiently large ℓ and m. In practice, we found that retaining the first 20×20 terms for the matrix $\Gamma_{\ell,m}(r)$ gave extremely accurate results (for a similar calculation see Swaters and Flierl (1991)).

All that remains to be determined in the exterior solution are the $\mathscr{K}_n$ coefficients in (3.2a) and (3.2c). Note that because of our choice of writing (3.2c) and our choice in determining α_ℓ in (3.6), it follows that $h_n|_{r=a} = \mathscr{K}_n \sin\theta$ for all n. Consequently, it follows from (2.17b), that

$$\sum_{n=0}^{\infty} \mathscr{K}_n \Pi_n(z) = aU(z), \tag{3.7a}$$

from which it follows that

$$\mathscr{K}_n = a \int_0^1 U(z) \Pi_n(z)\, dz, \tag{3.7b}$$

in which (2.16c) has been used. This completes the determination of the exterior solution.

In *non-wavelike* or *classical* modon theory, the interior modon wavenumber $\mu^{1/2}$ present in the solution (3.1) is determined by demanding that the azimuthal velocity given by φ_r, be continuous at $r = a$. In the modon solution presented here it is *not* possible to determine μ such that the azimuthal velocity is continuous across $r = a$ because of the presence of the $\{\sin(n\theta)\}_{n=1}^{\infty}$ terms in the exterior region which does not occur in the interior region and because of the sum over the vertical modes. In classical modon theory, there is only the single $\sin(\theta)$ term in the exterior and interior regions and the vertical structure is barotropic so that it is possible to select a modon wavenumber so that φ_r is continuous at the modon boundary. There is no contradiction with the underlying principles of inviscid fluid mechanics since, in the quasi-geostrophic limit, the continuity of the geostrophic streamfunction on the streamline defining the modon boundary ensures that leading order pressure and (trivially) the normal mass flux is continuous at $r = a$ for our solution. The continuity of the azimuthal velocity field on the modon boundary which is a streamline is not formally required in inviscid fluid mechanics. We will discuss the implications of this more completely next section.

There are two choices one can make about the modon wavenumber in the solution presented here. Either do nothing, and leave the modon wavenumber unspecified, free to take on empirically determined values as obtained from actual blocking data, or invoke some sort of closure ansatz. We have decided to do the latter realizing that this choice is arbitrary. Our choice is to mimic the classical solution as closely as possible and determine the modon wavenumber by insisting that the azimuthal velocity field associated with the $\sin(\theta)$ contribution of the gravest vertical mode be continuous at $r = a$. We may formally write this constraint in the form

$$\lim_{r\to a+} \int_0^1 \int_0^{2\pi} \Pi_0(z) \sin\theta p_r(r,\theta,z)\, d\theta\, dz = \lim_{r\to a-} \int_0^1 \int_0^{2\pi} \Pi_0(z) \sin\theta p_r(r,\theta,z)\, d\theta\, dz. \tag{3.8}$$

In the limit of a barotropic modon with no external Rossby wave-tail, it will be shown next section that (3.8) reduces to the barotropic limit of the Larichev and Reznik (1976) modon dispersion relationship. Finally, we point out that it is not possible to select a vertical dependence for μ so that the azimuthal velocity is continuous at the modon boundary.

Substituting (3.1) and (3.2) into (3.8) yields, after a little algebra, the baroclinic radiating modon dispersion relationship in the form

$$\frac{\alpha\gamma_0 J_2[(\mu-\lambda_0)^{1/2}a]}{(\mu-\lambda_0)^{1/2} J_1[(\mu-\lambda_0)^{1/2}a]} = -\mathscr{K}_0(-\lambda_0)^{1/2} \sum_{\ell=1}^{\infty} \alpha_\ell \{\delta_{\ell,1} Y_2[(-\lambda_0)^{1/2}a] + b_{\ell,1} J_2[(-\lambda_0)^{1/2}a]\}/Y_\ell[(-\lambda_0)^{1/2}a]. \tag{3.9}$$

This relationship implicitly defines $\mu = \mu(a, U, \beta)$ where the dependence of μ on the ambient baroclinic zonal flow $U(z)$ is manifested in the γ_0 and $\mathscr{K}_0$ coefficients. For a given set of parameters, there are a countable infinity of μ solutions satisfying $(\mu - \lambda_0) > 0$. The smallest value of $\mu^{1/2}$ such that $\mu > \lambda_0$ will be called the ground-state modon wavenumber and the corresponding modon called the ground-state baroclinic wave-like modon.

4. Dynamical characteristics and a simple example

The solution as constructed has continuous pressure and normal mass flux at the modon boundary. However, only the azimuthal velocity field associated with the $\sin(\theta)$ component of the gravest vertical mode is continuous at $r = a$. Consequently, all the other horizontal and vertical modes associated with φ_r will not be continuous at $r = a$. This implies that the potential vorticity is not continuous at $r = a$. But, as we now show, the interior and exterior limits of the potential vorticity at the modon boundary exists and are equal. Thus the limit of the potential vorticity at the modon boundary exists.

If we define the *total* potential vorticity to be given by $q \equiv \Delta p + (N^{-2}p_z)_z + \beta y$ where p is the *total* geostrophic pressure, it follows from (2.5a) and (2.9a) that

$$\lim_{r \to a^+} q = \lim_{r \to a^-} \Lambda_0(z)p = 0, \tag{4.1a}$$

$$\lim_{r \to a^-} q = \lim_{r \to a^-} [\Lambda_0(z) - \mu]p = 0, \tag{4.1b}$$

on account of (2.13) and (2.14), respectively. Hence we conclude

$$\lim_{r \to a} q = 0. \tag{4.2}$$

Because the limit exists, it follows that the jump in the potential vorticity across the modon boundary, defined by

$$[q]_a \equiv \lim_{r \to a^+} q - \lim_{r \to a^-} q,$$

is identically zero, that is $[q]_a = 0$. The wave-like baroclinic modon boundary, therefore, corresponds to a cylindrical vortex sheet with a zero potential vorticity jump.

The solution for the geostrophic pressure $p(x, y, z)$ as constructed above is therefore a classical solution of the potential vorticity equation except on the set of measure zero given by the cylinder $r = a$. Note that as $r \to a$, our solution satisfies (2.1a) since

$$\lim_{r \to a} J(p, q) = a^{-1}\{\lim_{r \to a} p_r q_\theta - \lim_{r \to a} p_\theta q_r\} = 0,$$

since $\text{limit}_{r \to a^{+,-}} |p_r, q_r| < \infty$ and $\text{limit}_{r \to a} p_\theta = \text{limit}_{r \to a} q_\theta = 0$.

The most important dynamical properties of the solution can be nicely illustrated with the following simple example. Suppose the fluid has constant stratification $N \equiv \text{constant} > 0$ (stable stratification) and the mean zonal flow is barotropic $U \equiv U_*$ (constant) satisfying $U_* > 0$ (an eastward flow). It follows from (2.7) that

$$\Lambda_0 = -\beta/U_* < 0, \tag{4.3}$$

since $\beta > 0$ and is obviously constant. The orthonormalized vertical modes $\Pi_n(z)$, determined by (2.16), are given by

$$\Pi_0(z) = 1, \tag{4.4a}$$

$$\Pi_n(z) = \sqrt{2}\cos(n\pi z), \tag{4.4b}$$

for $n = 1, 2, 3, \ldots$, with corresponding eigenvalues

$$\lambda_n = -(\beta/U_*) + n^2\pi^2/N^2. \tag{4.4c}$$

In particular, note that

$$\lambda_0 = -(\beta/U_*) < 0. \tag{4.4d}$$

The solution for the *total* interior streamfunction, determined by (2.24), (2.25b, c), (2.26) and (3.1), is simply

$$p = \Pi_0(z)g_0(x, y), \qquad \text{for } r < a, \tag{4.5}$$

since it follows from (2.25c) that $\gamma_n = -\beta\delta_{n,0}$. Substitution of (4.4) and (3.1) into (4.5) implies that the total interior streamfunction can be written in the form

$$p = -\beta\left\{\frac{aJ_1[(\mu-\lambda_0)^{1/2}r]}{(\lambda_0-\mu)J_1[(\mu-\lambda_0)^{1/2}]a} + r(\mu-\lambda_0)^{-1}\right\}\sin\theta, \tag{4.6}$$

in $r < a$.

The *total* exterior streamfunction, given by (2.2), (2.15), (2.17), (3.2), (3.4), (3.6) and (3.7) can be written in the form

$$p = -U_*y + \Pi_0(z)h_0(x, y), \tag{47}$$

since it follows from (3.7b) that $\mathscr{K}_n = aU_*\delta_{n,0}$. Substitution of (3.5) and (4.4) into (4.7) implies that the total exterior streamfunction can be written in the form

$$p = -U_*y + U_*a\sum_{m=1}^{\infty}\left\{\sum_{\ell=1}^{\infty}\alpha_\ell\Gamma_{\ell,m}(r)\right\}\sin(m\theta), \tag{4.8}$$

where $\Gamma_{\ell,m}(r)$ is determined by (3.5b) and (3.4) and the α_ℓ coefficients are determined by (3.6).

Finally, the modon dispersion relationship (3.9) can be written in the form

$$\frac{J_2[(\mu-\lambda_0)^{1/2}a]}{(\mu-\lambda_0)^{1/2}J_1[(\mu-\lambda_0)^{1/2}a]} = (-\lambda_0)^{-1/2}\sum_{\ell=1}^{\infty}\alpha_\ell\{\delta_{\ell,1}Y_2[(-\lambda_0)^{1/2}a] + b_{\ell,1}J_2[(-\lambda_0)^{1/2}a]\}/Y_\ell[(-\lambda_0)^{1/2}a]. \tag{4.9}$$

Clearly, this example corresponds to a stationary *barotropic* modon embedded in a constant barotropic eastward flow which will have a downstream

Rossby wave-tail. Upstream, however, the radiation condition (2.23) is satisfied and there will be no Rossby wave field. Although barotropic, this example will serve to illustrate the horizontal structure of the wave-like modon.

In this example we set $U_* > 0$. It is easy to see how in this example the Rossby wave-tail disappears if $U_* < 0$. The solution for the total interior streamfunction remains the same, albeit it is now understood $\lambda_0 = -(\beta/U_*) > 0$. However, if $\lambda_0 > 0$, then the solution for the total exterior streamfunction given by (4.8) dramatically changes. In this situation the Bessel functions in $\Gamma_{\ell,m}(r)$ are replaced by the modified $K_\ell[(\lambda_0)^{1/2}r]$ Bessel functions. However, these functions decay exponentially-rapidly at infinity so the radiation condition (2.22) is trivially satisfied. Consequently, $b_{\ell,m} \equiv 0$ for all (ℓ, m). The boundary condition (3.6) will therefore imply that the only remaining term in $\Gamma_{\ell,m}(r)$ will be proportional to $K_1[(\lambda_0)^{1/2}r]$. The dispersion relation is transformed accordingly with the second term on the right-hand-side of (4.9) identically zero. The resulting dispersion relationship is identical to the barotropic limit of the modon dispersion relationship derived by Larichev and Reznik (1976). It is the desire to have our solution reduce to the barotropic limit of the Larichev and Reznik dispersion relationship if $U_* < 0$ that motivates our choice for determining the baroclinic modon wavenumber via (3.8).

We will conclude this section by illustrating the example barotropic solution with the additional specific parameter values of $N = U_* = \beta = a = 1.0$. The ground-state modon wavenumber will be given approximately by $\mu^{1/2} \simeq 3.55$ (corresponding to $(\mu - \lambda_0)^{1/2} \simeq 3.69$). In Figs. 1 and 2 we show, respectively, a cross-section for $-2 < y < 2$ on $x = 0$ of the *total*

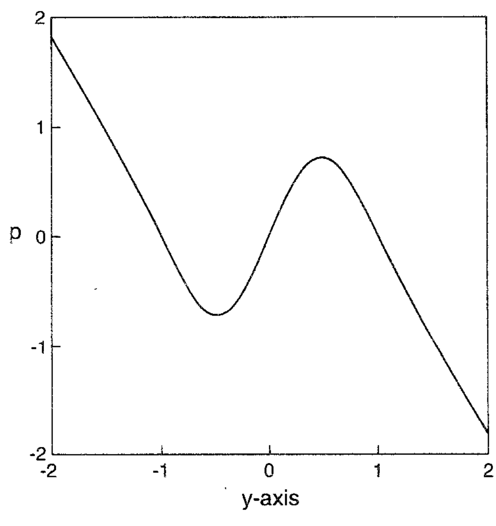

Figure 1
A y-cross-section of the total streamfunction field on $x = 0$ for $-2 \leq y \leq 2$. The modon boundary is located at $y = -1$ and $+1$, respectively.

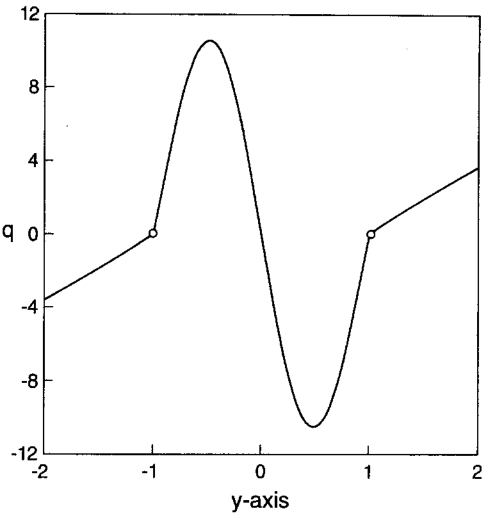

Figure 2
A y-cross-section of the total potential vorticity field on $x = 0$ for $-2 \leq y \leq 2$. The modon boundary, which is a vortex sheet in the solution presented here, corresponds to the two open circles located at $y = -1$ and $+1$, respectively.

streamfunction $p(x, y, z)$ and *total* potential vorticity $q = \Delta p + (N^{-2}p_z)_z + \beta y$ (appreciating, of course, that $p_z \equiv 0$ in this particular example). We remark that p and q are continuous but *not* differentiable at $r = a$ (located at $y = \pm 1$ where the two cusps occur in Fig. 2). The apparent near-smoothness at the modon boundary in the total streamfunction (see Fig. 1) is a simple numerical consequence of the fact that the radial derivatives associated with the higher azimuthal harmonics make a relatively small contribution to the overall solution in comparison to the sin(θ) mode near the modon boundary for these parameter values.

In Figs. 3a and 3b, we present "close-up" and "large-scale" contour plots of the total potential vorticity $\Delta p + (N^{-2}p_z)_z + \beta y$ for $-2 \leq x, y \leq 2$ and $-10 \leq x, y \leq 10$, respectively. The modon boundary corresponds to the closed circular O-contour. The reader is reminded that the zero value on the contour associated with the modon boundary represents the limit of the potential vorticity at the modon radius $r = a$. The potential vorticity at $r = a$ does not formally exist since the modon boundary corresponds to a vortex sheet. The Rossby wave-tail, which is not particularly evident in Fig. 3a is clearly seen in Fig. 3b. Note how the wave-tail is confined to the downstream region in accordance with the radiation condition (2.22). Also, as pointed out earlier, note that the potential vorticity field is an odd function with respect to y. This is the pattern observed in the numerical experiments reported by Haines and Marshall (1987).

In Figs. 4a and 4b we present the analogue "close-up" and "large-scale" contour plots of the total streamfunction $p(x, y, z)$ associated with the potential vorticity field shown in Fig. 3. Here, again, the modon boundary corresponds to the circular closed O-contour. The ambient flow moves eastward and the modon is stationary.

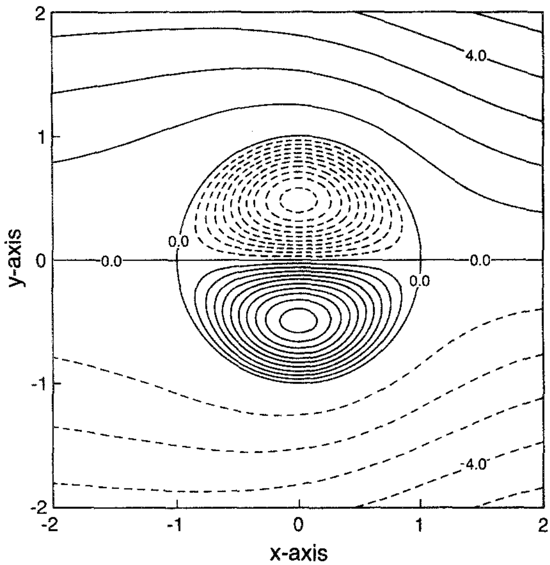

Figure 3a
A "close-up" contour plot of the potential field for $-2 \leq x, y \leq 2$. The solid and dashed contours correspond to non-negative and negative isolines of potential vorticity, respectively. The contour increment is about ± 1.0. The modon boundary corresponds to the circular zero-value contour. The maximum and minimum values of the potential vorticity within the modon interior are about $+10.5$ and -10.5, respectively.

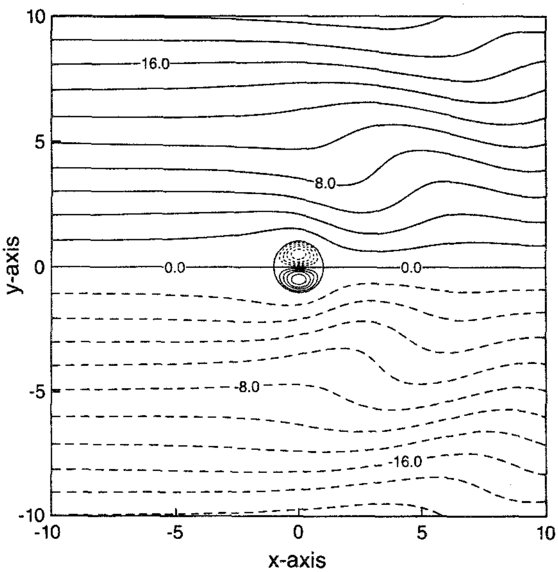

Figure 3b
A "large-scale" contour plot of the total potential vorticity field for $-10 \leq x, y \leq 10$. The solid and dashed contours are in Fig. 3a. The contour increment is about ± 2.0.

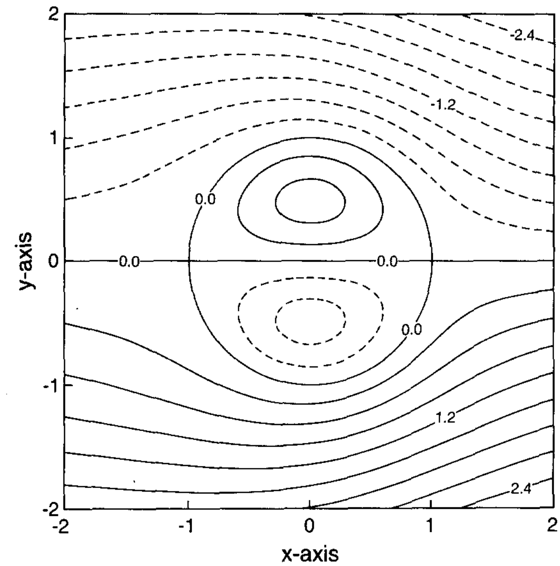

Figure 4a
A "close-up' contour plot of the total streamfunction field for $-2 \leq x, y \leq 2$. The solid and dashed lines correspond to non-negative and negative streamline values, respectively. The contour increment is about ± 0.3. The modon boundary corresponds to the circular zero-value contour. The relative maximum and minimum values of the streamfunction within the modon interior are about $+0.72$ and -0.72, respectively.

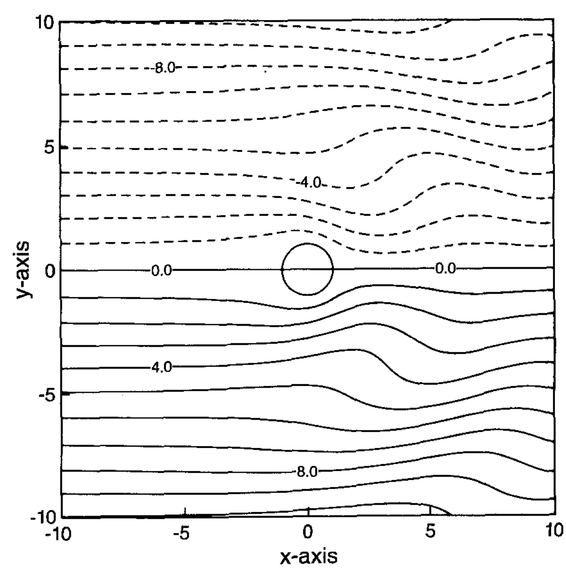

Figure 4b
A "large-scale" contour plot of the total streamfunction field for $-10 \leq x, y \leq 10$. The solid and dashed contours are as in Fig. 4a. The contour increment is about ± 1.0.

5. Summary

The structure of stationary, baroclinic modons embedded in a baroclinic eastward flow in a continuously-stratified fluid of finite depth has been determined. The gravest mode necessarily contains radiating Rossby waves in accordance with the radiation theorem established by Butchart et al. (1989). The solution we obtain satisfies the correct upstream radiation condition. The solution we constructed has continuous leading-order geostrophic pressure and normal-mass flux at the modon boundary. However, it is not possible to construct a radiating modon solution for which the azimuthal velocity field is continuous at the modon boundary. We have adopted the ansatz of determining the modon wavenumber by demanding that the azimuthal velocity field associated with the $\sin(\theta)$ component of the gravest vertical mode be continuous at the modon boundary. It was shown that this constraint reduces in the barotropic limit to the barotropic limit of the Larichev and Reznik (1976) modon dispersion relationship. Finally, we illustrated our solution with a simple barotropic example. Our solution has many qualitative features consistent with the numerical simulation of modons with a Rossby wave-tail presented by Haines and Marshall (1987).

There are several issues we have not addressed in this paper. We have not examined the stability of these solutions to infinitesimal perturbations. Nor have we examined how our solutions would be modulated with varying upstream flows, bottom topography, Ekman friction and so on. These and other issues must be examined before the role of radiating modons in geophysical fluid dynamics is to be understood.

The fact that we cannot choose the modon wavenumber in such a way as to ensure that the azimuthal velocity field is continuous at the modon boundary implies that the modon boundary in our solution is a vortex sheet. The implications of this property on the stability of the present solution needs to be further examined in order to better determine the physical importance of this solution as a model for a stationary dipole with a Rossby wave-tail on a β-plane. Preliminary numerical time integrations of the equivalent-barotropic potential vorticity equation using the stationary solution presented in Section 4 as an initial condition suggests that the qualitative features of the solution remain coherent for several eddy circulation time scales. However, it needs to be emphasized again these observations are preliminary and further study is required.

Another issue we have not examined here is the effect of the wave drag associated with the wave-tail. Because there will be a nonzero downstream momentum flux associated with the wave-tail, there must be an energy source for the waves. Consequently, there will be a decay in the strength of the dipole over time. The stationary ansatz introduced here will make sense only if the decay time scale is long in comparison with the eddy circulation time scale. However, it can be shown that the net or integrated quasi-geostrophic energy flux is zero. A detailed calculation of this decay based on the full primitive equations is required to verify that our ansatz has physical merit.

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Abstract

It is qualitatively known that the gravest mode associated with a vertical normal-mode decomposition of a stationary, baroclinic modon in a continuously-stratified fluid of finite depth on a β-plane necessarily contains an exterior downstream Rossby wave field if the background zonal flow is eastward. An exact solution is presented describing this situation which satisfies the correct upstream no-waves constraint.

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