Viscous modulation of the Lamb dipole vortex

G. E. Swaters
Applied Mathematics Institute, Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

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A simple analytical singular perturbation theory is developed to describe the viscous adiabatic decay of the two-dimensional Lamb dipole vortex. The vortex parameters (translation speed, radius, and wavenumber) evolve so as to satisfy leading-order globally averaged energy and enstrophy balances. These transport equations are shown to be solvability conditions for the asymptotic expansion. Extensions of the asymptotic procedure to other isolated vortex problems are discussed.

A ubiquitous feature in many experimental and numerical realizations of *two-dimensional* (2-D) turbulence is the emergence of coherent inertial dipole vortices as an important transport mechanism for the red energy and blue enstrophy cascades.[1–7] The advective interaction that occurs between two oppositely signed isolated vortices results in their mutual translation. Consequently, solitary vortex dipoles play a role in the homogenization and redistribution of energy, vorticity, and enstrophy in a turbulent 2-D flow (e.g., geostrophic turbulence). It is of interest, therefore, to develop simple analytical theories for the modulation and distortion of isolated coherent eddies as they propagate through a variable medium. The principal objective of this Letter is to present a simple perturbation theory that describes the adiabatic frictional dissipation of the Lamb (or Batchelor) dipole vortex,[8,9] assuming a relatively large, but finite Reynolds number. Also, we shall briefly discuss how the procedures developed here can be immediately exploited in other isolated eddy problems in fluid and plasma dynamics.

We begin with the scaled two-dimensional incompressible homogeneous Navier–Stokes equations written in the form

$$\Delta\varphi_t + J(\varphi,\Delta\varphi) = R^{-1}\Delta^2\varphi, \tag{1}$$

where φ is the streamfunction [with corresponding velocity field $\mathbf{u}\equiv(u,v)\equiv\mathbf{e}_3\times\nabla\varphi$], $\Delta\equiv\partial^2_{xx}+\partial^2_{yy}$, the Jacobian is given by $J(A,B)\equiv A_xB_y - B_xA_y$ (subscripts indicate differentiation), and R is the Reynolds number. Rotational invariance of (1) allows us to assume, with no loss of generality, that the x and y coordinate directions are parallel and transverse to the direction of vortex propagation, respectively.

In the inviscid limit (i.e., $R\rightarrow\infty$) an exact nonlinear steadily translating isolated eddy solution to (1) is the Lamb dipole vortex, which can be written in the form

$$\varphi(x,y,t) = a^2cr^{-1}\sin(\theta), \quad r>a, \tag{2a}$$

$$\varphi(x,y,t) = [2ck^{-1}J_1(kr)/J_0(ka) - cr]\sin\theta, \quad r<a, \tag{2b}$$

with the "dispersion" relation

$$J_1(ka) = 0, \tag{2c}$$

and where the comoving polar coordinates $r^2 \equiv (x-ct)^2 + y^2$ and $\tan(\theta)\equiv y/(x-ct)$ have been introduced. The parameter k is the eddy "wavenumber," c is the translation speed, and a is the eddy "boundary" between the *interior* ($r<a$) and the *exterior* ($r>a$) regions. The solution (2) has continuous pressure, velocities, and vorticity everywhere in $\mathbb{R}^2$, but has a finite discontinuity in the radial vorticity gradient across $r=a$. Note that $\varphi + cy = 0$ on $r=a$, so that the Lamb dipole eddy traps fluid particles in the interior region. The *ground-state* eddy dispersion relationship is given by

$$ka = j_{1,1}, \tag{3}$$

where $j_{1,1}\equiv 3.831\,71$ is the first nontrivial zero of the Bessel function $J_1(*)$.

The Lamb dipole is an example of the class of pseudo-three-dimensional solitary eddy solutions in which the vorticity can be expressed as a nonanalytic function of the comoving streamfunction. (The special feature of the Lamb dipole is that the flow in the *exterior* region is irrotational.) Solitary eddy dipoles that have nonanalytic functional relationships between the comoving streamfunction and the vorticity, but which possess nontrivial vorticity gradients in their exterior regions, have been called *modons.*[10] These solutions have been obtained for many models in geophysical fluid dynamics[11,12] and plasma physics.[13–19] We shall describe the application of the asymptotic procedures developed here to these other models toward the end of this Letter.

When friction is relatively weak in (1) (i.e., $0<R^{-1}\ll 1$), it is possible to obtain an adiabatic perturbation solution for the dissipating Lamb dipole vortex as follows. We introduce the rapidly varying phase variables

$$\xi\equiv x - R\int_0^{t/R} c(t')dt', \tag{4a}$$

$$y\equiv y, \tag{4b}$$

and the *slow* time variable

$$T\equiv t/R. \tag{4c}$$

Substitution of (4) into the Navier–Stokes equations (1) gives

$$J(\varphi + cy,\Delta\varphi) = R^{-1}\Delta^2\varphi - R^{-1}\Delta\varphi_T, \tag{5}$$

where the Jacobian is now given by $J(A,B) = A_\xi B_g - A_yB_\xi$, and $\Delta\equiv\partial^2_{\xi\xi}+\partial^2_{yy}$. We solve (5) with a straightforward asymptotic expansion of the form

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$$\varphi \sim \varphi^{(0)}(\xi,y;T) + R^{1}\varphi^{(1)}(\xi,y;T) + R^{-2}\varphi^{(2)}(\xi,y;T) + \cdots . \quad (6)$$

The $O(1)$ problem is given by

$$J(\varphi^{(0)} + cy, \Delta\varphi^{(0)}) = 0, \quad (7)$$

for which we take as the solution the Lamb dipole (2) with the obvious modifications in the definitions of the comoving polar coordinates. The adiabatic ansatz is that $c \equiv c(T)$, $a \equiv a(T)$, and $k \equiv k(T)$ such that the dispersion relationship (2c) remains continuously satisfied. This approximation will hold until the vorticity amplitude has decreased to such a point that further dissipation is essentially a balance between the local time rate of change of vorticity and the viscous term in (1), i.e., a 2 + 1 heat equation for the vorticity. The dispersion relation forms a single constraint on the evolution of the three eddy parameters c, a, and k. Consequently, two additional transport equations are required in order to determine the evolution of the eddy parameters. These evolution equations are obtained as solvability conditions on the $O(R^{-1})$ problem.

The $O(R^{-1})$ problem can be put into the form

$$J(\varphi^{(0)} + cy, \Delta\varphi^{(1)} + \lambda\varphi^{(1)}) = \Delta^2\varphi^{(0)} - \Delta\varphi_T^{(0)}, \quad (8)$$

where $\lambda \equiv k^2$ in $r<a$ and $\lambda \equiv 0$ in $r>a$. The homogeneous adjoint problem associated with (8) can be written in the form

$$(\Delta + \lambda) J(\varphi^{(0)} + cy, q) = 0, \quad (9a)$$

for which there are two obvious solutions:

$$q \equiv \varphi^{(0)}, \quad (9b)$$

$$q \equiv \Delta\varphi^{(0)}. \quad (9c)$$

Therefore the inhomogeneity in (8) must satisfy the orthogonality conditions

$$\partial_T \int\int_{\mathbf{R}^2} (\Delta\varphi^{(0)})^2 = 2\int\int_{\mathbf{R}^2} \Delta^2\varphi^{(0)}\Delta\varphi^{(0)}, \quad (10a)$$

$$\partial_T \int\int_{\mathbf{R}^2} \varphi^{(0)}\Delta\varphi^{(0)} = 2\int\int_{\mathbf{R}^2} \varphi^{(0)}\Delta^2\varphi^{(0)}. \quad (10b)$$

The integrals in the transport equations (10) can be evaluated to yield, respectively,

$$c_T = -(j_{1,1})^2 c/a^2, \quad (11a)$$

$$(ac)_T = -(j_{1,1})^2 c/a, \quad (11b)$$

where the ground-state dispersion relationship has been used. It readily follows from (11) and the dispersion relation (2c) that

$$a_T \equiv k_T \equiv 0. \quad (12)$$

Thus the ground-state solutions for a, c, and k are simply

$$a(T) \equiv a_0, \quad (13a)$$

$$k(T) \equiv k_0 \equiv j_{1,1}/a_0, \quad (13b)$$

$$c(T) \equiv c_0 \exp(-k_0^2 T), \quad (13c)$$

where the zero subscript denotes the value at $T \equiv 0$.

The transport equations (10) have a simple physical interpretation. Equations (10a) and (10b) correspond to the leading-order globally averaged *enstrophy* and *energy* balance equations, respectively, for (1). The solutions (13) predict that during the period of time that the adiabatic ansatz can be made, the translation speed of the Lamb dipole will approach zero exponentially rapidly. Because the streamfunction and vorticity fields are proportional to $c(T)$ it immediately follows that the absolute amplitude of the Lamb dipole will also exponentially approach zero. The asymptotic procedures developed in this Letter are two-dimensional generalizations of the direct perturbation methods that have been developed to describe Rayleigh-perturbed one-dimensional solitary wave equations, assuming a soliton or solitary wave initial condition.[19]

We have applied the above asymptotic analysis to the frictional dissipation problem for the barotropic *modon* solutions of the Charney–Hasegawa–Mima equation[20,21] and compared the results with high-resolution numerical simulations. The predictions of the theory are in very good agreement with the numerical solutions. One interesting difference between the dissipating modon and the theory developed here for the Lamb dipole is that in the modon we find that the vortex pair dilates during the decay process. We attribute this difference to the fact that the modon has nontrivial vorticity gradients in the exterior region because of the presence of a dispersive term in the vorticity evolution equation. This property results in a parametric coupling between the translation speed, radius, and wavenumber in the dispersion relationship and, consequently, in the adiabatic evolution of the modon parameters. We have also been able to apply these methods to study the evolution of topographically modulated modon solutions to the shallow-water equations.[22] We expect therefore that these methods will be useful in other perturbed solitary eddy problems.

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Analysis of energy transfer in direct numerical simulations of isotropic turbulence

J. Andrzej Domaradzki
Department of Aerospace Engineering, University of Southern California, University Park, Los Angeles, California 90089-1191

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The energy transfer between wavenumber bands is calculated from the results of direct numerical simulations of isotropic, decaying turbulence. The results are consistent with the notion of the energy cascade induced by the interactions among wavenumbers of comparable lengths. The nonlocal interactions are also present and give rise to the inverse energy cascade.

At the present time there is controversy concerning the importance of local versus nonlocal interactions in turbulent flows. Analysis of experimental data performed by Deissler[1] for isotropic turbulence and by Lii *et al.*[2] for high Reynolds number turbulence in planetary boundary layers points toward a large degree of nonlocalness. Also the results of Kraichnan[3,4] obtained from turbulence closures indicate that about 25% of the energy transfer in turbulent flows is due to nonlocal wavenumber triads with the ratio of the shortest to the middle leg greater than 2. Even a larger degree of nonlocalness in the energy transfer is suggested by Dannevik *et al.*[5] through a concept of local beltramization of the strong turbulence.

On the other hand, the classical phenomenology of Kolmogorov stresses local interactions (energy cascade) as a leading cause of the universal inertial subrange. Also, the numerical work of Kerr,[6] Brasseur and Corrsin,[7] and Domaradzki *et al.*[8] for low Reynolds number turbulence indicates that very little energy is transferred between distant wavenumbers, whereas similar wavenumbers transfer energy in a manner consistent with the concept of the local energy cascade. Domaradzki *et al.*[8] also raised the intriguing question of the possibility of nonlocal inverse energy transfer (from small to large scales). Such a possibility was predicted by Sivashinsky and Yakhot[9] and Shtilman and Sivashinsky[10] for anisotropic flows, but it is an unexpected feature of isotropic turbulence. The purpose of this work is to investigate in detail the above problems using numerically generated turbulent fields.

The quantity of principal interest in this Letter is the energy exchange $T_R(\mathbf{k})$ between a given mode $\mathbf{k}$ and all pairs of modes $\mathbf{p},\mathbf{k}-\mathbf{p}$ that form a triangle within $\mathbf{k}$ as one of the legs, and such that at least one of them ($\mathbf{p}$ or $\mathbf{k}-\mathbf{p}$) lies in the prescribed region $\mathbb{R}$ in the wavenumber space. In addition to the velocity field $u_n(\mathbf{k})$ given on the entire mesh (we omit explicit time dependence in all formulas) we define a truncated velocity field

$$u'_n(\mathbf{k}) = \begin{cases} u_n(\mathbf{k}), & \text{for } \mathbf{k}\in\mathbb{R}, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The energy transfer $T_R(\mathbf{k})$ may be calculated from the Navier–Stokes equations and is given by the following formula:

$$T_R(\mathbf{k}) = 2\Big[\mathrm{Im}\Big(u_n^*(\mathbf{k})P_{new}(\mathbf{k})\int d^3p\, u'_e(\mathbf{p})u_w(\mathbf{k}-\mathbf{p})\Big) - \frac{1}{2}\mathrm{Im}\Big(u_n^*(\mathbf{k})P_{new}(\mathbf{k})\int d^3p\, u'_e(\mathbf{p})u'_w(\mathbf{k}-\mathbf{p})\Big)\Big], \quad (2)$$

where

$$P_{new}(\mathbf{k}) = k_w(\delta_{ne} - k_nk_e/k^2) + k_e(\delta_{nw} - k_nk_w/k^2), \quad (3)$$

an asterisk denotes complex conjugate, and the summation convention is assumed. For isotropic fields it is convenient to choose the region $\mathbb{R}$ as a shell in the wavenumber space $a<|\mathbf{k}|<b$ and to investigate the averaged energy transfer

$$T(k;a,b) = 4\pi k^2 \tfrac{1}{2}\langle T_R(\mathbf{k})\rangle, \quad (4)$$

where $\langle\cdots\rangle$ denotes averaging over thin spherical shells of radius k. Note that if $\mathbb{R}$ covers the entire wavenumber space $0\leqslant|\mathbf{k}|\leqslant k_{\max}$, then $T(k;0,k_{\max})$ is the usual energy transfer modeled in the turbulence closures. In the first term in (2), when integrating over $\mathbf{p}$, contributions from triads with both $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$ belonging to $\mathbb{R}$ are included twice and contributions from triads with only $\mathbf{p}$ in $\mathbb{R}$ are counted once. The

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