Perturbations of Soliton Solutions to the Unstable Nonlinear Schrödinger and Sine-Gordon Equations

By *Gordon E. Swaters*

The adiabatic evolution of soliton solutions to the *unstable* nonlinear Schrödinger (UNS) and sine-Gordon (SG) equations in the presence of small perturbations is reconsidered. The transport equations describing the evolution of the solitary wave parameters are determined by a direct multiple-scale asymptotic expansion and by phase-averaged conservation relations for an arbitrary perturbation. The evolution associated with a dissipative perturbation is explicitly determined and the first-order perturbation fields are also obtained.

1. Introdution

The unstable nonlinear Schrödinger (UNS) and sine-Gordon (SG) equations arise as canonical integrable models describing the weakly nonlinear evolution of the disturbance field in marginally stable or unstable oceanographic and meteorological dynamics [1–5] (and in many other dispersive physical systems, e.g., [6]). In their canonical form, these equations model the space-time development of wave packets assuming that there is no variability in the fluid medium or in the mean flow or that nonconservative processes are

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present. It is known, however, that time variability in the background flow and dissipative processes can have a profound effect on the linear and nonlinear stability characteristics of atmospheric and ocean currents [7–11]. Indeed, even if the time average of the background flow is itself stable, small amplitude oscillations can lead to linear destabilization or vice versa (even if the oscillatory flow is, at each moment in time, linearly stable or unstable, respectively).

In the context of modeling the finite amplitude evolution of marginally stable or unstable baroclinic flow with, for example, time variability and/or dissipation, one is naturally led to the perturbed UNS or SG equations (see, e.g., [3,5,9–11]). Given the generic emergence and persistence of isolated coherent structures in the transition to turbulence in oceanographic and meteorological dynamics [12], it is of interest to understand the dynamic consequences of perturbations on the soliton solutions of the UNS and SG equations as these represent the saturated states of these models.

Huang et al. [13] have presented a theory for the adiabatic deformation of the solitary wave solution to the UNS equation based on the inverse scattering formalism similar to that developed for integrable equations by Kaup and Newell [14] and catalogued for numerous other soliton models by Kivshar and Malomed [15]. The principal purpose of this paper is to reconsider the perturbed UNS equation via a direct perturbation expansion and phase-averaged "conservation balances" following the methods described by, for example, Kodama and Ablowitz [16] and Grimshaw [17, 18], respectively. Our results from both the direct perturbation expansion and conservation balance approaches, of course, completely agree with each other, but contradict Huang et al. [13]. Moreover, in the context of the dissipative problem, we are able to explicitly determine the first-order perturbation field.

Additionally, but on a more minor note, we have identified an error in the treatment of the perturbed SG equation in [16]. The result of this error is that the equation describing the adiabatic evolution of the translation velocity for the kink solitary wave solution of the perturbed SG equation in [16] is incorrect and does not agree with the inverse scattering results in [14] or [15]. We hasten to add, however, that the overall conceptual approach via a direct asymptotic expansion to perturbed solitary wave equations presented in [16] is correct. There is simply a relatively minor algebraic error in the section on the perturbed SG equation that leads to an incorrect evolution equation for the translation velocity and, additionally, for the solution given for the perturbation field. We briefly present the correct direct approach to the perturbed SG equation for a general perturbation and explicitly solve describe the first-order perturbation field for the dissipative problem.

2. The unstable nonlinear Schrödinger equation

We consider the perturbed (nondimensional) UNS equation in the form

$$v_{tt} + i v_x + 2v|v|^2 = \varepsilon F(\varepsilon t, v), \quad -\infty < x < \infty, \quad t > 0, \tag{1}$$

where $(x,\ t)$ are the space-time coordinates, respectively, $v(x,\ t)$ is the complex-valued dependent variable, $F(\varepsilon t,\ v) \simeq O(1)$ is the complex-valued perturbation that explicitly depends only on v (and its derivatives) and the "slow time" εt, where $0 < \varepsilon << 1$ (and $i^2 = -1$). The UNS equation is a special case of the Ginzburg–Landau equation and is of the form of the nonlinear Schrödinger equation (NLS) with x and t interchanged [e.g., 19]. Because the UNS equation is second order in time, the initial conditions for the Cauchy problem require data for v and v_t, which differs from NLS that is only single order in time. The inverse scattering transform for the UNS equation was described by Yajima and Wadati [20].

When $\varepsilon = 0$, the soliton solution to (1) can be written in the form

$$\begin{aligned} v_{\text{soliton}}(x,t) = \\ \mu \ \text{sech}\left[\frac{\mu(x - x_0 - Ut)}{U}\right] \times \exp\left[i \left\{ \left(\mu^2 - \frac{1}{4U^2}\right)(x - x_0 - Ut) \right.\right. \\ \left.\left. + U\left(\mu^2 + \frac{1}{4U^2}\right)(t - t_0) \right\}\right], \end{aligned} \tag{2}$$

where μ, U, and $(x_0,\ t_0)$ are arbitrary real-valued amplitude, translation velocity, and space-time phase-shift parameters, respectively. Our goal here is to determine the adiabatic evolution of the soliton when $0 < \varepsilon << 1$, that is, to determine the leading order solution to (1) assuming

$$v(x,0) = v_{\text{soliton}}(x,0) \quad \text{and} \quad v_t(x,0) = \partial_t v_{\text{soliton}}(x,0), \tag{3}$$

to leading order.

2.1. Direct singular perturbation expansion

Here, we determine the leading-order adiabatic evolution of the soliton by constructing a direct perturbation expansion for (1), assuming $0 < \varepsilon << 1$, allowing the soliton parameters to slowly evolve over time following the method generally outlined in [16]. The evolution of the soliton parameters is determined by applying appropriate solvability conditions, which are that the inhomogeneous terms in the higher order problems must be orthogonal to the Kernel of the adjoint operator associated with the linear first order perturbation problem.

We begin by introducing the fast phase and slow time variables

$$\theta = x - \frac{1}{\varepsilon}\int_0^{\varepsilon t} U(\xi)\, d\xi \quad \text{and} \quad T = \varepsilon t, \tag{4}$$

respectively, and introducing

$$\Phi \equiv \frac{1}{\varepsilon}\int_0^{\varepsilon t} U(\xi)\left[\mu^2(\xi) + \frac{1}{4U^2(\xi)}\right] d\xi, \tag{5}$$

so that

$$\theta_t = -U(T), \qquad \Phi_t = U(T)\left[\mu^2(T) + \frac{1}{4U^2(T)}\right], \tag{6}$$

$$\partial_x \to \partial_\theta \text{ and } \partial_{tt} \to U^2\partial_{\theta\theta} - \varepsilon\left[U\partial_{\theta T} + (U\partial_\theta)_T\right] + \varepsilon^2\partial_{TT}. \tag{7}$$

In terms of θ and T, the soliton solution (2) can be written in the form

$$v_{\text{soliton}} = \mu \ \operatorname{sech}(\mu\theta/U)\exp\left[i\left\{\left(\mu^2 - \frac{1}{4U^2}\right)\theta + \Phi\right\}\right]. \tag{8}$$

In (8) we have assumed that $x_0 = t_0 = 0$. As in the perturbed NLS or KdV problems (see, e.g., [14,16]), it can be shown that the leading order solvability conditions do not determine the evolution of the phase shift parameters and thus, without loss of generality for the analysis presented here, we may set them to zero. Their evolution is determined by higher order solvability conditions, or equivalently, higher order fast phase averaged conservation relations, that do not concern us here.

The adiabatically deformed soliton solution to (1) is most conveniently obtained in the form

$$v(\theta, T) = q(\theta, T)\exp\left[i\left\{\left(\mu^2 - \frac{1}{4U^2}\right)\theta + \Phi\right\}\right], \tag{9}$$

where $q(\theta,\ T)$ is complex-valued. Substitution of (4), (5), and (9) into (1) leads to, after some algebra,

$$\begin{aligned}
&\left(U^2\partial_{\theta\theta} - \mu^2\right)q + 2|q|^2 q \\
&\quad = \varepsilon\left\{\exp(-i\Psi)F(T, q\exp(i\Psi)) - iq_T/U + (Uq_\theta)_T + Uq_{\theta T}\right. \\
&\qquad + 2i\theta q_\theta U\left(\mu^2 - \frac{1}{4U^2}\right)_T + iq\left[U\left(\mu^2 - \frac{1}{4U^2}\right)_T + \frac{U_T}{2U^2}\right] \\
&\qquad \left. + (\theta q/U)\left(\mu^2 - \frac{1}{4U^2}\right)_T\right\} + O(\varepsilon^2),
\end{aligned} \tag{10}$$

where, for convenience,

$$\Psi \equiv \left(\mu^2 - \frac{1}{4U^2}\right)\theta + \Phi. \tag{11}$$

On account of the fact that the leading order solution will be real-valued, further progress is facilitated by substituting

$$q(\theta, T) = \eta(\theta, T) + i\varepsilon\phi(\theta, T), \tag{12}$$

where η and ϕ are real-valued into (10) and separating out the real and imaginary parts, which lead to, respectively,

$$\left(U^2\partial_{\theta\theta} - \mu^2\right)\eta + 2\eta^3 = \varepsilon\Upsilon_R(\eta, T) + O(\varepsilon^2), \tag{13}$$

$$\left(U^2\partial_{\theta\theta} - \mu^2 + 2\eta^2\right)\phi = \Upsilon_I(\eta, T) + O(\varepsilon), \tag{14}$$

where

$$\Upsilon_R(\eta, T) = \mathrm{Re}\,\{\exp(-i\Psi)F\,[T, \eta\exp(i\Psi)]\} + (U\eta_\theta)_T + U\eta_{\theta T} + (\theta\eta/U)\left(\mu^2 - \frac{1}{4U^2}\right)_T, \tag{15}$$

$$\Upsilon_I(\eta, T) = \mathrm{Im}\{\exp(-i\Psi)F\,[T, \eta\exp(i\Psi)]\} - \eta_T/U + 2\theta\eta_\theta U\left(\mu^2 - \frac{1}{4U^2}\right)_T + \eta\left[U\left(\mu^2 - \frac{1}{4U^2}\right)_T + \frac{U_T}{2U^2}\right]. \tag{16}$$

Introduction of the straightforward expansion

$$(\eta, \phi) \simeq (\eta, \phi)^{(0)} + \varepsilon(\eta, \phi)^{(1)} + \varepsilon^2(\eta, \phi)^{(2)} + \ldots,$$

into (13) and (14) leads to the $O(1)$ problems, respectively,

$$\left(U^2\partial_{\theta\theta} - \mu^2\right)\eta^{(0)} + 2\left[\eta^{(0)}\right]^3 = 0, \tag{17}$$

$$\left(U^2\partial_{\theta\theta} - \mu^2 + 2\left[\eta^{(0)}\right]^2\right)\phi^{(0)} = \Upsilon_I\left(\eta^{(0)}, T\right), \tag{18}$$

and the $O(\varepsilon)$ problem associated with (13) (which is all that is needed) is

$$\left(U^2\partial_{\theta\theta} - \mu^2 + 6\left[\eta^{(0)}\right]^2\right)\eta^{(1)} = \Upsilon_R\left(\eta^{(0)}, T\right). \tag{19}$$

From (17) the solution is simply

$$\eta^{(0)}(\theta, T) = \mu\ \mathrm{sech}(\mu\theta/U). \tag{20}$$

Observing that the operators on the left-hand-side of (18) and (19) are self-adjoint and that

$$\left(U^2\partial_{\theta\theta} - \mu^2 + 2\left[\eta^{(0)}\right]^2\right)\eta^{(0)} = 0, \tag{21}$$

$$\left(U^2\partial_{\theta\theta} - \mu^2 + 6\left[\eta^{(0)}\right]^2\right)\eta_\theta^{(0)} = 0, \tag{22}$$

implies that, necessarily,

$$\int_{-\infty}^{\infty} \eta^{(0)}\Upsilon_I\left(\eta^{(0)}, T\right) d\theta = 0, \tag{23}$$

$$\int_{-\infty}^{\infty} \eta_\theta^{(0)}\Upsilon_R\left(\eta^{(0)}, T\right) d\theta = 0, \tag{24}$$

which can be evaluated to yield, respectively,

$$\frac{d}{dT}\left(\frac{1}{U}\int_{-\infty}^{\infty}\left[\eta^{(0)}\right]^2 d\theta\right)$$
$$= 2\int_{-\infty}^{\infty}\eta^{(0)}\mathrm{Im}\left\{\exp(-i\Psi)F\left[T, \eta^{(0)}\exp(i\Psi)\right]\right\} d\theta, \tag{25}$$

$$\frac{d}{dT}\left(U\int_{-\infty}^{\infty}\left[\eta_\theta^{(0)}\right]^2 d\theta\right) - \frac{1}{2U}\left(\mu^2 - \frac{1}{4U^2}\right)_T\int_{-\infty}^{\infty}\left[\eta^{(0)}\right]^2 d\theta$$
$$= -\int_{-\infty}^{\infty}\eta_\theta^{(0)}\,\mathrm{Re}\left\{\exp(-i\Psi)F\left[T, \eta^{(0)}\exp(i\Psi)\right]\right\} d\theta, \tag{26}$$

which, if (20) is substituted in, can be further simplified to, respectively,

$$\mu_T = U\int_{-\infty}^{\infty}\ \mathrm{sech}(\xi)\ \mathrm{Im}\{\exp(-i\Psi)F[T, \mu\mathrm{sech}(\xi)\exp(i\Psi)]\}\, d\xi, \tag{27}$$

$$U_T = -2U^3\int_{-\infty}^{\infty}\ \mathrm{sech}(\xi)\tanh(\xi)\ \ \mathrm{Re}\{\exp(-i\Psi)F[T, \mu\mathrm{sech}(\xi)\exp(i\Psi)]\}\, d\xi, \tag{28}$$

where it is understood that $\Psi = U\xi[\mu^2 - 1/(4U^2)]/\mu + \Phi$ in the integrands in (27) and (28), see (11).

Equations (27) and (28) are the general transport equations describing the evolution of the UNS soliton amplitude and translation velocity for an arbitrary perturbation. Owing to the fact that we have set up the asymptotic expansion in a somewhat different manner than in [13], and more like that presented in [16] for the NLS equation, it is not possible to easily and directly compare (27) and (28) with corresponding expressions in [13].

If $F\,[T, \mu\mathrm{sech}(\xi)\exp(i\Psi)]$ is an even function with respect to ξ, then the integrand in (28) will be odd, and the right-hand-side of (28) will be zero. Thus, to leading order, the translation velocity of the UNS soliton will be invariant and only the amplitude will evolve with respect to T. This is qualitatively similar to the known results for the perturbed NLS equation, see [16]. Indeed, this is the case for the dissipative perturbation example explicitly solved for later in this section. The results of this specific example can be directly compared with those in [13]. As we then show, the results so obtained do not agree with those in Huang et al. [13].

2.2. Phase-averaged conservation relation approach

The transport equations (27) and (28) were derived using a straightforward perturbation expansion and applying appropriate solvability conditions. It is also possible to obtain (27) and (28) using, in our opinion, the more physically intuitive approach of phase averaged conservation relations like that described by Grimshaw [17,18]. The demonstration that (27) and (28) are obtained using this second approach gives confidence in the results since the transport equations derived here for the perturbed UNS equation do not agree with those in [13].

The "first three" conservation relations associated with (1) are given by

$$\begin{aligned}\left(v^* v_t - v v_t^*\right)_t + i(|v|^2)_x &= \varepsilon[v^* F(\varepsilon t, v) - v F^*(\varepsilon t, v^*)] \\ &= 2\varepsilon i\ \mathrm{Im}[v^* F(\varepsilon t, v)], \end{aligned} \tag{29}$$

$$\begin{aligned}\left(v_x^* v_t + v_x v_t^*\right)_t + \left(|v|^4 - |v_t|^2\right)_x &= \varepsilon\left[v_x^* F(\varepsilon t, v) + v_x F^*(\varepsilon t, v^*)\right] \\ &= 2\varepsilon \mathrm{Re}\left[v_x^* F(\varepsilon t, v)\right], \end{aligned} \tag{30}$$

$$\begin{aligned}&\left(|v_t|^2 + |v|^4 + \frac{i}{2}\left(v^* v_x - v v_x^*\right)\right)_t + \frac{i}{2}\left(v v_t^* - v^* v_t\right)_x \\ &= \varepsilon\left[v_t^* F(\varepsilon t, v) + v_t F^*(\varepsilon t, v^*)\right] = 2\varepsilon \mathrm{Re}\left[v_t^* F(\varepsilon t, v)\right], \end{aligned} \tag{31}$$

where $(\cdot)^*$ is the complex-conjugate of $(\cdot)$. Equations (29), (30), and (31) correspond to the mass, momentum, and energy balance relations, respectively, and are identical in form to those for the NLS equation with x and t interchanged, see [18]. Equations (29), (30), and (31) are obtained by forming $v^* \times (1) - v \times (1)^*$, $v_x^* \times (1) + v_x \times (1)^*$, $v_t^* \times (1) + v_t \times (1)^*$, respectively.

Generically, these conservation relations are of the form

$$E_t + H_x = \varepsilon G, \tag{32}$$

with E, F, and G the appropriate density, flux, and source term, respectively. If the multiple scale *ansatz* (4), together with the asymptotic expansion

$$v \simeq v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \dots,$$

is substituted into (32), one obtains an expression of the form

$$(-U\partial_\theta + \varepsilon\partial_T)\left(E^{(0)} + \varepsilon E^{(1)} + \dots\right) + \left(H^{(0)} + \varepsilon H^{(1)} + \dots\right)_\theta = \varepsilon\left(G^{(0)} + \cdots\right), \tag{33}$$

which if integrated with respect to θ (assuming E, H, and G all vanish sufficiently rapidly as $\theta \to \pm\infty$) yields the leading order phase-averaged conservation relation

$$\frac{d}{dT}\int_{-\infty}^{\infty} E^{(0)}\,d\theta = \int_{-\infty}^{\infty} G^{(0)}\,d\theta. \tag{34}$$

Since it follows from (8) that

$$v_t \simeq -Uv_\theta^{(0)} + i\Phi_t v^{(0)} + O(\varepsilon) = -Uv_\theta^{(0)} + iUv^{(0)}\left[\mu^2 + \frac{1}{4U^2}\right] + O(\varepsilon), \tag{35}$$

$$v_x \simeq v_\theta^{(0)} + O(\varepsilon), \tag{36}$$

where

$$v^{(0)} \equiv v_{\text{soliton}} = \eta^{(0)}\exp(i\Psi),$$

we find for the mass, momentum, and energy densities, after a little algebra,

$$E_{\text{mass}}^{(0)} = \left(v^* v_t - v v_t^*\right)^{(0)} = \frac{i\left[\eta^{(0)}\right]^2}{U}, \tag{37}$$

$$\begin{aligned} E_{\text{momentum}}^{(0)} &= \left(v_x^* v_t + v_x v_t^*\right)^{(0)} = -2U\left|v_\theta^{(0)}\right|^2 + i\Phi_t\left(v^{(0)}v_\theta^{(0)*} - v^{(0)*}v_\theta^{(0)}\right) \\ &= -2U\left[\eta_\theta^{(0)}\right]^2 + \left(\mu^2 - \frac{1}{4U^2}\right)\frac{\left[\eta^{(0)}\right]^2}{U}, \end{aligned} \tag{38}$$

$$E_{\text{energy}}^{(0)} = \left[|v_t|^2 + |v|^4 + \frac{i}{2}\left(v^* v_x - v v_x^*\right)\right]^{(0)} = \frac{\left[\eta^{(0)}\right]^2}{2U^2}, \tag{39}$$

respectively. Similarly, the leading order source (the G) terms associated with the mass, momentum, and energy conservation relations are given by, respectively,

$$G_{\text{mass}}^{(0)} = 2i\ \text{Im}\left[(v^* F(\varepsilon t, v))^0\right] = 2i\eta^{(0)}\text{Im}\left\{\exp(-i\Psi)F\left[\varepsilon t, \eta^{(0)}\exp(i\Psi)\right]\right\}, \tag{40}$$

$$\begin{aligned} G_{\text{momentum}}^{(0)} &= 2\ \text{Re}\left[\left(v_x^* F(\varepsilon t, v)\right)^{(0)}\right] \\ &= 2\ \text{Re}\left\{\left[\eta^{(0)}\exp(-i\Psi)\right]_\theta F\left[\varepsilon t, \eta^{(0)}\exp(i\Psi)\right]\right\}, \end{aligned} \tag{41}$$

$$
\begin{aligned}
G^{(0)}_{\text{energy}} &= 2\ \mathrm{Re}\big[\big(v_t^{*}F(\varepsilon t, v)\big)^{(0)}\big] \\
&= 2\ \mathrm{Re}\big\{\big(-U\big[\eta^{(0)}\exp(-i\Psi)\big]_\theta \\
&\quad - i\Phi_t\eta^{(0)}\exp(-i\Psi)\big)F\big[\varepsilon t, \eta^{(0)}\exp(i\Psi)\big]\big\}.
\end{aligned} \tag{42}
$$

It can be seen immediately that the phased-averaged mass conservation relation (which follows from (29), (34), and (37)) is identical to (25). The phased-averaged momentum conservation relation (which follows from (30), (34), and (38)) is given by

$$
\begin{aligned}
&\frac{d}{dT}\int_{-\infty}^{\infty}\left[U\big[\eta_\theta^{(0)}\big]^2 - \left(\mu^2 - \frac{1}{4U^2}\right)\frac{\big[\eta^{(0)}\big]^2}{2U}\right]d\theta \\
&= -\int_{-\infty}^{\infty}\mathrm{Re}\big\{\big[\eta^{(0)}\exp(-i\Psi)\big]_\theta F\big[\varepsilon t, \eta^{(0)}\exp(i\Psi)\big]\big\}\,d\theta.
\end{aligned} \tag{43}
$$

Equation (43) can be put exactly into the form of (26) if one notes that the second term in the integrand in the left-hand-side of (43) can be re-arranged as

$$
\begin{aligned}
&\frac{d}{dT}\int_{-\infty}^{\infty}\left(\mu^2 - \frac{1}{4U^2}\right)\frac{\big[\eta^{(0)}\big]^2}{2U}\,d\theta \\
&= \frac{1}{2U}\left(\mu^2 - \frac{1}{4U^2}\right)_T\int_{-\infty}^{\infty}\big[\eta^{(0)}\big]^2 d\theta + \frac{1}{2}\left(\mu^2 - \frac{1}{4U^2}\right)\frac{d}{dT}\int_{-\infty}^{\infty}\frac{\big[\eta^{(0)}\big]^2}{U}\,d\theta \\
&= \frac{1}{2U}\left(\mu^2 - \frac{1}{4U^2}\right)_T\int_{-\infty}^{\infty}\big[\eta^{(0)}\big]^2 d\theta \\
&\quad + \left(\mu^2 - \frac{1}{4U^2}\right)\int_{-\infty}^{\infty}\eta^{(0)}\ \mathrm{Im}\left\{\exp(-i\Psi)F\left[T, \eta^{(0)}\exp(i\Psi)\right]\right\}\,d\theta \\
&= \frac{1}{2U}\left(\mu^2 - \frac{1}{4U^2}\right)_T\int_{-\infty}^{\infty}\big[\eta^{(0)}\big]^2 d\theta \\
&\quad + \int_{-\infty}^{\infty}\eta^{(0)}\ \mathrm{Im}\left\{i\,[\exp(-i\Psi)]_\theta\, F\left[T, \eta^{(0)}\exp(i\Psi)\right]\right\}\,d\theta \\
&= \frac{1}{2U}\left(\mu^2 - \frac{1}{4U^2}\right)_T\int_{-\infty}^{\infty}\big[\eta^{(0)}\big]^2 d\theta \\
&\quad + \int_{-\infty}^{\infty}\eta^{(0)}\ \mathrm{Re}\left\{[\exp(-i\Psi)]_\theta\, F\left[T, \eta^{(0)}\exp(i\Psi)\right]\right\}\,d\theta,
\end{aligned}
$$

where (25) has been used and the fact that $\mathrm{Im}\,[i(a+ib)] = \mathrm{Re}(a+ib)\,\forall\, a, b \in \mathbb{R}$, so that (43) can be put into the form

$$\frac{d}{dT}\left(U\int_{-\infty}^{\infty}\left[\eta_{\theta}^{(0)}\right]^{2}d\theta\right)-\frac{1}{2U}\left(\mu^{2}-\frac{1}{4U^{2}}\right)_{T}\int_{-\infty}^{\infty}\left[\eta^{(0)}\right]^{2}d\theta$$
$$=-\int_{-\infty}^{\infty}\mathrm{Re}\left\{\left[\eta^{(0)}\exp(-i\Psi)\right]_{\theta}F\left[\varepsilon t,\eta^{(0)}\exp(i\Psi)\right]\right\}d\theta$$
$$+\int_{-\infty}^{\infty}\eta^{(0)}\mathrm{Re}\left\{\left[\exp(-i\Psi)\right]_{\theta}F\left[T,\eta^{(0)}\exp(i\Psi)\right]\right\}d\theta$$
$$=-\int_{-\infty}^{\infty}\eta_{\theta}^{(0)}\mathrm{Re}\left\{\exp(-i\Psi)F\left[T,\eta^{(0)}\exp(i\Psi)\right]\right\}d\theta,$$

which is exactly (26).

In addition, although we do not show it here, it follows that the phase-averaged mass and momentum conservation relations imply that the phase-averaged energy conservation relation (as obtained from (31), (34), and (38)) will be satisfied. Thus, in summary, from two different approaches we have derived a consistent set of transport equations for the adiabatic deformation of a perturbed UNS soliton.

2.3. Example with a dissipative perturbation

To provide an explicit example that can be compared directly with the results of Huang et al. [13] and with the known results for the perturbed NLS equation [14–16], we consider the dissipative perturbation

$$F(\varepsilon t, v) = -i\gamma(\varepsilon t)v, \tag{44}$$

in (1), where $\gamma(\varepsilon t) > 0$ is a prescribed $O(1)$ coefficient function. Substitution of (44) into (27) and (28) yields, respectively,

$$\mu_T = -U\gamma(T)\mu\int_{-\infty}^{\infty}\mathrm{sech}^2(\xi)\,d\xi = -2U\gamma(T)\mu, \tag{45}$$

$$U_T = 0, \tag{46}$$

which have the solutions

$$\mu(T) = \mu_0\exp\left[-2U_0\int_0^T\gamma(\xi)\,d\xi,\right] \tag{47}$$

$$U(T) = U_0, \tag{48}$$

where $\mu_0 = \mu(0)$ and $U_0 = U(0)$, respectively. We see that the soliton amplitude will decay exponentially over time and the translation velocity is constant.

Huang et al.'s [13] treatment of the dissipative perturbation (44) assumes $\gamma \equiv 1$ (and they write the soliton parameters in a slightly, but inconsequential, different way). In terms of our soliton parameters, the prediction in [13] is that

$$\mu_T = -\frac{U\mu}{1 + 4U^2\mu^2} \quad \text{and} \quad U_T = 0,$$

thus there is agreement for the prediction of invariance in the translation velocity but a difference between our and their prediction for the rate of decrease in the soliton amplitude (set $\gamma = 1$ in (45)). It is beyond the scope of this paper to fully dissect the many algebraic manipulations in [13] to find the precise error(s). The fact that we have obtained (47) and (48) through two different derivation paths gives confidence that ours is the correct result.

2.3.1. *First-order perturbation field.* We can determine the first order perturbation fields $\eta^{(1)}$ and $\phi^{(0)}$ as follows. For the dissipative perturbation (44), assuming the transport or solvability conditions (45) and (46), it follows from (18) and (19), after a little algebra, that

$$\left[\partial_{\zeta\zeta} - 1 + 2\text{sech}^2(\zeta)\right]\phi^{(0)} = \gamma\mu^{-1}(1 - 4U^2\mu^2)\ \text{sech}(\zeta)[1 - 2\zeta\ \tanh(\zeta)], \quad (49)$$

$$\left[\partial_{\zeta\zeta} - 1 + 6\ \text{sech}^2(\zeta)\right]\eta^{(1)} = 8\gamma U\text{sech}(\zeta)\tanh(\zeta)[1 - \zeta\ \tanh(\zeta)], \quad (50)$$

respectively, where we have introduced the change of independent variable $\zeta = \mu\theta/U$. Note that $\zeta \to +\infty$ will always correspond to the far field *ahead* of the propagating soliton irrespective of the sign of U.

If we introduce the change of dependent variables

$$\phi^{(0)} = \text{sech}(\zeta)\tilde{\phi}(\zeta) \quad \text{and} \quad \eta^{(1)} = \text{sech}(\zeta)\tanh(\zeta)\tilde{\eta}(\zeta),$$

into (49) and (50), it follows that

$$\begin{aligned}\left[\text{sech}^2(\zeta)\tilde{\phi}_\zeta\right]_\zeta &= \gamma\mu^{-1}(1 - 4U^2\mu^2)\text{sech}^2(\zeta)\,[1 - 2\zeta\ \tanh(\zeta)] \\ &= \gamma\mu^{-1}(1 - 4U^2\mu^2)\left[\zeta\,\text{sech}^2(\zeta)\right]_\zeta, \end{aligned} \quad (51)$$

$$\left[\text{sech}^2(\zeta)\tanh^2(\zeta)\tilde{\eta}_\zeta\right]_\zeta = 8\gamma U\text{sech}^2(\zeta)\tanh^2(\zeta)\,[1 - \zeta\ \tanh(\zeta)], \quad (52)$$

which, individually, can be integrated twice to yield

$$\tilde{\phi} = \frac{\gamma(1 - 4U^2\mu^2)\zeta^2}{2\mu},$$

$$\tilde{\eta} = 2\gamma U\zeta\,[\zeta - \coth(\zeta)],$$

so that, in terms of the original fast phase variable θ, we obtain

$$\phi^{(0)}(\theta, T) = \frac{\gamma\mu(1 - 4U^2\mu^2)\theta^2\text{sech}(\mu\theta/U)}{2U^2}, \quad (53)$$

$$\eta^{(1)}(\theta, T) = 2\gamma\mu\theta \operatorname{sech}(\mu\theta/U)\left[\frac{\mu\theta}{U}\tanh(\mu\theta/U) - 1\right]. \tag{54}$$

Finally, correct to $O(\varepsilon)$, the adiabatically dissipating UNS soliton will be described by

$$v(x,t) \simeq \left\{\mu \operatorname{sech}(\mu\theta/U) + \varepsilon\left[\eta^{(1)} + i\phi^{(0)}\right]\right\} \exp\left[i\left\{\left(\mu^2 - \frac{1}{4U^2}\right)\theta + \Phi\right\}\right], \tag{55}$$

where $\theta(x, \varepsilon t)$, $\Phi(\varepsilon t)$, $\mu(\varepsilon t)$, $U(\varepsilon t)$, $\phi^{(0)}(\theta, \varepsilon t)$, and $\eta^{(1)}(\theta, \varepsilon t)$ will be given by (4), (5), (47), (48), (53), and (54), respectively. As in the dissipatively perturbed NLS soliton [16,18] there is no "shelf region" that emerges in the $O(\varepsilon)$ solution in the far field *behind* the propagating soliton, i.e., $|\eta^{(1)} + i\phi^{(0)}| \to 0$ exponentially rapidly as $\theta \to -\text{sgn}(c)\infty$. This is a reflection of the fact that adiabatic dissipating soliton is simultaneously satisfying phase-averaged mass, energy, and momentum conservation relations. However, clearly the asymptotic solution (55) is algebraically nonuniform as $\theta \to \pm\infty$. This algebraic nonuniformity in the far field can be eliminated by introducing asymptotic techniques of the form described in [16] and is beyond the scope of this paper.

3. The sine-Gordon equation

Because the UNS equation is second order in time, we thought it would be useful to compare the results of the theory just presented with the direct perturbation theory, as described in [16], for the sine-Gordon (SG) equation, which is also second order in time. Unfortunately, there is an easily identified minor algebraic error in the implementation of the multiple-scale *ansatz* in [16] that results in the derivation of an incorrect transport equation for the soliton–translation velocity and the solution of the perturbation field for the perturbed SG equation. As written, in fact, the results in [16] do not agree with the results, based on inverse scattering theory, as described in [14]. We hasten to add, however, that the overall conceptual description of the direct perturbation theory in [16] is completely correct.

Here, we very briefly present the corrected direct perturbation theory. Our results will agree with those in [14]. In laboratory coordinates, the perturbed (nondimensional) SG equation can be written in the form

$$v_{tt} - v_{xx} + \sin(v) = \varepsilon F(\varepsilon t, v), \tag{56}$$

where $(x,\ t)$ are the space-time coordinates, respectively, $v(x,\ t)$ is the real-valued dependent variable, $F(\varepsilon t,\ v) \simeq O(1)$ is the real-valued perturbation that explicitly depends only on v and the "slow time" εt, where $0 < \varepsilon << 1$. In the limit $\varepsilon = 0$, (56) has the kink soliton solution

$$v_{\text{soliton}} = 4\tan^{-1}\left[\exp(x - ct - x_0)/\sqrt{1-c^2}\right], \tag{57}$$

where x_0 and c are the arbitrary phase shift and translation velocity parameters (note that $|c| < 1$), respectively. As before, the leading order solvability conditions do not determine the evolution of the phase shift parameter and thus, without loss of generality for the analysis presented here, we henceforth assume $x_0 = 0$.

To determine the adiabatic deformation of the kink soliton, we introduce the fast phase and slow time variables

$$\theta = x - \frac{1}{\varepsilon}\int_0^{\varepsilon t} c(\xi)\,d\xi \quad \text{and} \quad T = \varepsilon t. \tag{58}$$

It follows that

$$\partial_{xx} \to \partial_{\theta\theta}, \tag{59}$$

and that

$$\partial_{tt} \to (-c\partial_\theta + \partial_T)(-c\partial_\theta + \partial_T) = c^2\partial_{\theta\theta} - \varepsilon[c\partial_{\theta T} + (c\partial_\theta)_T] + \varepsilon^2\partial_{TT}. \tag{60}$$

It is at this point that there is an error in Section 3.2 in [16]. In the transition from [16]'s equation (3.18) to their equation (3.19), Kodama and Ablowitz [16] have assumed that

$$\partial_{tt} \to c^2\partial_{\theta\theta} + 2\varepsilon(c\partial_\theta)_T + O(\varepsilon^2),$$

rather than the correct form in (60). This is a minor algebraic error, but it leads to erroneous conclusions about the evolution of $c(T)$ and the solution for the first-order perturbation field. We will see that with (60), the associated solvability conditions will exactly reproduce the deformation theory obtained from inverse scattering theory [14].

Substitution of (58) into (56) leads to

$$(c^2 - 1)v_{\theta\theta} + \sin(v) = \varepsilon\,[cv_{\theta T} + (cv_\theta)_T + F(T, v)] + O(\varepsilon^2). \tag{61}$$

Introducing the straight forward asymptotic expansion

$$v \simeq v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \dots, \tag{62}$$

into (61) leads to the $O(1)$ and $O(\varepsilon)$ problems, respectively,

$$(c^2 - 1)v_{\theta\theta}^{(0)} + \sin\left(v^{(0)}\right) = 0, \tag{63}$$

$$\left[(c^2 - 1)\partial_{\theta\theta} + \cos\left(v^{(0)}\right)\right]v^{(1)} = cv_{\theta T}^{(0)} + \left(cv_\theta^{(0)}\right)_T + F\left(T, v^{(0)}\right). \tag{64}$$

The solution to (63) is taken to be the kink soliton written in the form

$$v^{(0)} = 4\tan^{-1}\left[\exp(\theta/\sqrt{1-c^2})\right]. \tag{65}$$

Observing that the operator on the left-hand-side of (64) is self-adjoint and that a homogeneous solution is given by

$$v^{(1)}_{\text{homogeneous}} = v^{(0)}_{\theta} = \frac{2\operatorname{sech}(\theta/\sqrt{1-c^2})}{\sqrt{1-c^2}}, \tag{66}$$

which implies that the right-hand-side of (64) must satisfy

$$\frac{d}{dT}\left(\int_{-\infty}^{\infty} c\left[v^{(0)}_{\theta}\right]^2 d\theta\right) = -\int_{-\infty}^{\infty} F\left(T, v^{(0)}\right) v^{(0)}_{\theta}\, d\theta. \tag{67}$$

If we appreciate that

$$\int_{-\infty}^{\infty} [v^{(0)}_{\theta}]^2\, d\theta = \frac{8}{\sqrt{1-c^2}},$$

then it is seen that (67) exactly reproduces the results for the perturbed SG equation in Section 7 in [14].

Alternatively, (67) can be obtained from the phase-averaged momentum conservation relation obtained by multiplying (56) through by v_x and writing the result in the form

$$(v_t v_x)_t - \left[\cos(v) + \frac{v_t^2 + v_x^2}{2}\right]_x = \varepsilon v_x F(\varepsilon t, v). \tag{68}$$

Hence, if we introduce the variables (58) and the expansion (62), we see that (67) follows after integrating (68) with to $\theta \in (-\infty, \infty)$.

3.1. Example with a dissipative perturbation

To explicitly compare with the calculation in [16], we assume the dissipative perturbation

$$F(\varepsilon t, v) = -\gamma(\varepsilon t) v_t,$$

in (56). It follows from (67) that

$$\frac{d}{dT}\left(c\int_{-\infty}^{\infty}\left[v^{(0)}_{\theta}\right]^2 d\theta\right) = -\gamma c\int_{-\infty}^{\infty}\left[v^{(0)}_{\theta}\right]^2 d\theta,$$

which can be evaluated to give

$$(c/\sqrt{1-c^2})_T = -\gamma c/\sqrt{1-c^2} \Longrightarrow c_T = -\gamma c(1-c^2), \tag{69}$$

which is different than the result in [16] (see the result immediately following equation (3.21) in [16]). Equation (69) can be immediately integrated to yield

$$c(T) = \frac{c_0}{\sqrt{c_0^2 + (1 - c_0^2)\exp\left[2\int_0^T \gamma(\xi)\,d\xi\right]}}, \tag{70}$$

and we note that $1 - c_0^2 > 0$.

3.1.1. *First order perturbation field*. We can determine the solution for $v^{(1)}$ directly as follows. From [64], we have

$$\begin{aligned}\left[(c^2 - 1)\partial_{\theta\theta} + \cos\left(v^{(0)}\right)\right]v^{(1)} &= cv_{\theta T}^{(0)} + \left(cv_\theta^{(0)}\right)_T + c\gamma v_\theta^{(0)} \\ &= -\gamma c^3\left(v_\theta^{(0)} + 2\theta v_{\theta\theta}^{(0)}\right). \end{aligned}\tag{71}$$

If we define $v^{(1)} = v_\theta^{(0)}\tilde{v}$, it follows that

$$\left(\left[v_\theta^{(0)}\right]^2 \tilde{v}_\theta\right)_\theta = \frac{\gamma c^3}{1 - c^2}\left(\theta\left[v_\theta^{(0)}\right]^2\right)_\theta,$$

which can be integrated twice to imply

$$v^{(1)} = \frac{\gamma c^3\theta^2 \operatorname{sech}(\theta/\sqrt{1 - c^2})}{(1 - c^2)^{\frac{3}{2}}}, \tag{72}$$

which is different than equation (3.23) in [16]. As in [16], we find that there is no shelf region formed behind the propagating soliton. However, the solution for $v^{(1)}$ is algebraically nonuniform in the far field. This nonuniformity can be removed using appropriate asymptotic techniques.

4. Conclusions

We have presented an adiabatic perturbation theory for the soliton solution of the unstable nonlinear Schrödinger (UNS) equation. The theory was constructed using both a direct perturbation approach, like that described in [16] for a number of other 1 + 1 dimensional soliton models, and fast-phase averaged conservation relations, like that described in [17,18]. Our results, while internally consistent, disagree with the results of Huang et al. [13] for the UNS equation. General transport equations were derived for the soliton amplitude and translation velocity for an arbitrary perturbation. These transport equations were solved in the case of a dissipative perturbation in order to explicitly compare our results with those in [13]. We find, as in [13], that the translation velocity is invariant (analogous to that which occurs in the NLS equation with damping, see [16]). However, we find a much simpler decay result for the soliton amplitude than in [13].

In addition, we have determined the first order perturbation field for the dissipative example. Again, as in perturbed NLS theory, we find that no shelf region emerges behind the propagating soliton. This occurs because the transport equations for the soliton parameters simultaneously satisfy phase-averaged energy, momentum, and mass conservation relations.

Because the UNS equation is second order in time, it was thought useful to compare the results of our analysis with the direct perturbation theory for the sine-Gordon (SG) equation, which is also second order in time. Unfortunately, we have identified a minor algebraic error in the multiple scale asymptotic theory described in [16] for the SG equation. As written the results in [16] do not agree with the known results from inverse scattering theory [14]. We briefly describe the corrected direct perturbation theory for the SG equation in laboratory coordinates and explicitly solve the problem for a dissipative perturbation and in order compare our results with those in [16].

The original motive for this paper resides in our desire to analyze the finite amplitude development of baroclinically unstable flows with time variability and dissipation present in the so-called “frontal geostrophic” dynamical regime [21–23]. This is a dynamical regime that describes stratified sub-inertial ocean currents with large amplitude isopycnal defections on length scales longer than the internal deformation radius. In the weakly nonlinear finite-amplitude limit, it is known that marginally unstable waves will satisfy the UNS equation (5) or the SG equation (2). When time variability and dissipation is included, the resulting wave-packet equations are of the form of the perturbed UNS equation. Clearly, it is important from the viewpoint of the oceanographic application that we are interested in to properly understand the deformation characteristics associated with these soliton models when perturbations are present. Once it became clear that the results in [13] were not reproducible, it was thought appropriate to reconsider this problem in a suitably general way from both the point of view of the direct perturbation approach and phased-averaged conservation relations. Our results on the oceanographic problem will be published in a venue better oriented to that audience.

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