A HAMILTONIAN STRUCTURE FOR HYPERELASTIC FLUID-FILLED TUBES

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ABSTRACT. The governing equations describing a teth-
ered hyperelastic fluid-filled tube are shown to possess a non-
canonical Hamiltonian formulation. The Hamiltonian struc-
ture is exploited to give a variational principle for finite-
amplitude steadily-travelling solitary and periodic pressure
pulses. Sufficient conditions for the linear and nonlinear sta-
bility in the sense of Liapunov are described for these solu-
tions.

1. Introduction. The study of waves in elastic fluid-filled tubes is of interest particularly in regards to, among others, blood flow and transmission devices in spacecraft. From the viewpoint of mechanics, the subject is of current interest because it represents a coupling between two branches of continuum mechanics: nonlinear hydrodynamics and elasticity. There is a substantial literature, both theoretical and applied, on waves in fluid-filled elastic tubes. While a thorough review is beyond the scope of this article, most studies have tended to focus either on the dispersive aspects ignoring nonlinearity (e.g., Rubinow and Keller [25, 26] and Moodie, et al. [18, 19]), or on the nonlinear aspects ignoring dispersion (e.g., Moodie and Haddow [17], Anliker, et al. [1] and Seymour and Mortell [29]).

It is only in the last ten years or so that the *combined* effects of nonlinearity and dispersion in the fluid and elastic wall have been examined. For example, Cowley [10, 11] based on an earlier theory of Moodie and Haddow [17] showed that, as expected, the evolution of long weakly-nonlinear elastic jumps in hyperelastic fluid-filled tubes are governed by a KdV equation. Moodie and Swaters [20] extended the Moodie and Haddow theory to examine shock formation in tubes with variable wall thicknesses. Swaters and Sawatzky [31] extended the Cowley model to include viscoelastic effects in the tube wall in order to model the pulse attenuation and broadening that is experimentally observed (e.g., Caro et al., [8]) for solitary pressure pulses. Swaters [30]

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developed a theory for wave-wave interactions in hyperelastic fluid-filled tubes in an attempt to model energy transfers within wave spectra.

Recently, Ropchan and Swaters [24] have examined the shear-flow instability problem for hyperelastic fluid-filled tubes. In addition to the usual linear Rayleigh instability of a homogeneous fluid (modified, of course, by the presence of the elastic wall), Ropchan and Swaters found that otherwise neutrally stable modes could resonantly interact and produce explosive instabilities which become unbounded in finite time. It is of interest, therefore, to develop a unified theory of the finite-amplitude dynamics of fluid-filled tubes. The principal purpose of this paper is to develop a Hamiltonian description of the dynamics of tethered hyperelastic fluid-filled tubes and to exploit the Hamiltonian formalism to discuss aspects of the dynamical characteristics of steadily-travelling solutions.

The plan of this paper is as follows. In Section 2 the basic model and boundary conditions are introduced. In Section 3 the Hamiltonian structure is introduced and the Casimir and impulse invariants are found.

In Section 4 the governing equations describing fully nonlinear periodic and solitary steadily-travelling solutions are derived. It is shown that in the infinitesimally small amplitude limit, this equation yields that known dispersion relation for linear dispersive pressure pulses in tethered hyperelastic fluid-filled tubes. It is also shown that under an appropriate low wavenumber weakly nonlinear scaling, this equation yields small-but-finite amplitude soliton and periodic cnoidal wave solutions. In addition, in Section 4, we derive a variational principle for the fully nonlinear steadily-travelling solutions in terms of a suitably constrained Hamiltonian.

In Section 5 conditions are determined for the positive definiteness of the second variation of the constrained Hamiltonian evaluated at the steadily-travelling solution. This result is exploited to establish the linear stability in the sense of Liapunov of the steadily-travelling solution. Based on the linear stability analysis, appropriate convexity hypotheses are introduced on the constrained Hamiltonian which can, in principle, establish the nonlinear stability of the steadily-travelling solutions in the sense of Liapunov.

2. Governing equations. Since the derivation of the basic model is well known, we will be brief in our presentation. The nonlinear *dimensional* equations governing the fluid are given by the mass conservation equation and axial momentum equation given by, respectively,

$$(2.1) \qquad A_{t^*}^* + (u^* A^*)_{x^*} = 0,$$

$$(2.2) \qquad (\partial_{t^*} + u^* \partial_{x^*}) u^* + \frac{1}{\rho} p_{x^*}^* = 0,$$

where ρ, $A^*(x^*, t^*)$, $u^*(x^*, t^*)$ and $p^*(x^*, t^*)$ are the constant fluid density, cross-sectional area, axial fluid velocity and fluid pressure, respectively. The axial coordinate is given by x^* and t^* is time. Throughout our work it will be assumed that $x^* \in R \subset \mathbf{R}$ and $t^* > 0$. Subscripts with respect to (x^*, t^*) indicate the appropriate partial derivative unless otherwise stated.

It is important to point out that these equations implicitly assume that the flow is radially symmetric and that the radial variations of the axial velocity may be neglected. This latter approximation is physically reasonable if the radial accelerations are small in comparison, on average, to the axial accelerations in the fluid and is analogous to the usual assumptions of classical shallow-water theory. This does *not* imply, however, that there is no radial motion; rather these are expressed through changes in the cross-sectional area.

To close (2.1) and (2.2) an additional relationship is required between, say, the pressure and the cross-sectional area. It can be shown for axisymmetric homogeneous membranous nonlinear elastic shells (e.g., Green and Zern [14] or Ogden [21]); the relevant elastic theory for the present context is described by Moodie and Haddow [17], that the dimensional pressure drop across the tube wall can be expressed in the form

$$(2.3) \quad p^*(x^*, t^*) = \frac{H}{a^*(1+e)} \frac{\partial W^*}{\partial \lambda_1} - \frac{H}{a^*} \frac{\partial}{\partial x^*} \left\{ \frac{a_0 a_{x^*}^*}{[1 + (a_{x^*}^*)^2]^{1/2}} \frac{\partial W^*}{\partial \lambda_2} \right\},$$

where H, a^*, a_0 and e are the wall thickness, time-dependent wall radius, reference wall radius and the imposed axial pre-strain due to the tethering force (which prevents axial motion in the wall), and where the *strain-energy function* is given by

$$(2.4) \qquad W^* = W^*(\lambda_1, \lambda_2),$$

where λ_1 and λ_2 are the azimuthal and axial principal stretches given by, respectively,

$$(2.5) \qquad \lambda_1 \equiv a^*/a_0,$$

$$(2.6) \qquad \lambda_2 \equiv (1+e)[1 + (a_{x^*}^*)^2]^{1/2}.$$

The cross-sectional area is, of course, related to the wall radius by

$$(2.7) \qquad A^* = \pi(a^*)^2.$$

It is important to point out that inertial effects in the tube have been neglected in the derivation of (2.3). Cowley [10] has shown that this approximation is valid even for finite-amplitude deformations provided

$$\frac{\rho_w H}{\rho a_0} \ll 1,$$

where ρ_w is the density of the material comprising the tube wall. Another point to make is that the pressure drop across the tube wall *must* take the form (2.3) if the mechanical properties of the tube wall are to be based on a *rational* theory of finite-amplitude elasticity. In addition, the well-known mathematical and physical restrictions (see, e.g., Ogden [21]) on strain energy functions for isotropic Green materials must hold. These conditions will further restrict the allowed functions $W^*(\lambda_1, \lambda_2)$. We will state these properties as they are needed.

It is convenient in what follows to nondimensionalize the governing equations. To this end we introduce the nondimensional (nonasterisked) variables

$$(2.8) \qquad x^* = a_0 x, \qquad t^* = \frac{a_0}{U_0}t, \qquad U_0 \equiv \left[\frac{HW_0}{a_0\rho(1+e)}\right]^{1/2},$$

$$(2.9)$$
$$u^*(x^*, t^*) = U_0 u(x, t), \qquad p^*(x^*, t^*) = \rho U_0^2 p(x, t),$$

$$(2.10)$$
$$a^*(x^*, t^*) = a_0 a(x, t), \qquad A^*(x^*, t^*) = \pi a_0^2 A(x, t),$$

$$(2.11)$$
$$W(a, (1+e)\lambda) \equiv W^*(\lambda_1, \lambda_2)/W_0,$$

$$(2.12) \qquad \lambda \equiv (1 + a_x^2)^{1/2}.$$

Substituting (2.8) through to (2.12) into (2.1) through to (2.7) leads to the *nondimensional* system

(2.13)
$$(\partial_t + u\partial_x)u + p_x = 0,$$

(2.14) $\quad\quad A_t + (Au)_x = 0,$

(2.15)
$$p = \frac{1}{a}\frac{\partial W}{\partial a} - \frac{1}{a}\frac{\partial}{\partial x}\left[\frac{a_x}{(1+a_x^2)^{1/2}}\frac{\partial W}{\partial \lambda}\right],$$

(2.16) $\quad\quad A = a^2,$

where it is understood that $W(a, (1+e)\lambda)$ and λ are defined by (2.11) and (2.12), respectively.

We will consider, in general, deformations about the uniformly pre-stressed state with constant pressure $p_\infty \neq 0$ (since $e > 0$). This uniform state must correspond to a steady solution of the model, i.e.,

(2.17)
$$p_\infty = W_a^\infty,$$

where

(2.18)
$$W_a^\infty \equiv \left[\frac{\partial W}{\partial a}\right]_{(a,\lambda)=(1,1)}.$$

One physical situation of interest corresponds to solutions which decay at infinity to the pre-stressed configuration. The appropriate boundary conditions in this case are, respectively,

(2.19) $\quad\quad p \to p_\infty,$

(2.20) $\quad\quad a \to 1,$

(2.21) $\quad\quad u \to 0,$

as $|x| \to \infty$ and it is understood that $R = \mathbf{R}$. Note that because (2.13) and (2.14) are invariant under Galilean transformations, if it is assumed that u approaches a constant value as $|x| \to \infty$, we may, without loss of generality, set that constant equal to zero. Solutions which satisfy (2.19) to (2.21) will be said to be solitary.

It is also of interest to examine spatially periodic solutions in a uniformly prestressed tube. In this situation the appropriate boundary conditions are given by, respectively,

(2.22) $p(0) = p(L),$

(2.23) $a(0) = a(L),$

(2.24) $u(0) = u(L),$

where the domain is now given by $R = \{x \mid 0 < x < L\}$. Geometrically, this configuration corresponds to a deformable torus.

Another configuration of interest are steadily-travelling elastic jumps. These solutions are similar to hydraulic jumps in shallow water theory [10]. The boundary conditions associated with these solutions may be written in the form

$$
(2.25) \qquad \lim_{x \to \infty} \begin{pmatrix} a \\ p \\ u \end{pmatrix} = \begin{pmatrix} a^+ \\ p^+ \\ u^+ \end{pmatrix},
$$

$$
(2.26) \qquad \lim_{x \to -\infty} \begin{pmatrix} a \\ p \\ u \end{pmatrix} = \begin{pmatrix} a^- \\ p^- \\ u^- \end{pmatrix},
$$

where

$$
(2.27) \qquad p^+ = \left[\frac{\partial W}{\partial a} \right]_{(a,\lambda)=(a^+,1)},
$$

$$
(2.28) \qquad p^- = \left[\frac{\partial W}{\partial a} \right]_{(a,\lambda)=(a^-,1)},
$$

and it is assumed that $(a^+, p^+, u^+) \neq (a^-, p^-, u^-)$.

3. Hamiltonian structure for fluid-filled hyperelastic tubes.

3.1. *Hamiltonian formulation.* Over the last several years, Hamiltonian formulations of various fluid mechanical equations have been used to make very general and deep observations on the structure and stability of fluid flows. For example, modern solitary wave theory draws heavily on the Hamiltonian structure of the governing equations. In

particular, soliton stability theory (e.g., [5, 7]) and soliton modulation theory (e.g., [17]) directly exploit the Hamiltonian structure associated with the governing equations. Arnol'd's hydrodynamic stability theorems [2, 3, 4, 15] are also best understood as a consequence of the Hamiltonian structure of the governing equations.

In many situations of physical interest, a system of partial differential equations does not have a least action principle associated with it in terms of the desired dependent variables. For example, while the Euler equations have a variational principle in terms of Lagrangian variables (e.g., [32, 27]), they do *not* have a variational principle in terms of the Eulerian pressure and velocity [28] without having to introduce a Clebsch transformation. It is, however, possible to *define* a Hamiltonian formulation of a general system of partial differential equations as a mathematical formulation which satisfies the algebraic properties associated with a classical symplectic Hamiltonian system as described, for example, by Goldstein [13]. The definition given here closely parallels the approach taken by, for example, Benjamin [6] or Olver [22].

Definition 1. A system of n partial differential equations, written in the form

$$(3.1) \qquad \mathcal{F}\left(\mathbf{q}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right) = 0,$$

where t is time and $\mathbf{q}(\mathbf{x}, t) = [q_1(\mathbf{x}, t), \dots, q_n(\mathbf{x}, t)]^T$ is a column vector of n dependent variables with the m independent spatial variables $\mathbf{x} = (x_1, \dots, x_m)$ defined on the open spatial domain $\Omega \subseteq \mathbf{R}^m$ with the boundary (if it exists) $\partial\Omega$, is said to be *Hamiltonian* if there exists a conserved functional $H(\mathbf{q})$, called the *Hamiltonian*, and a matrix $\mathbf{J}$ of (possibly pseudo) differential operators such that (3.1) can be written in the form

$$(3.2) \qquad \mathbf{q}_t = \mathbf{J}\frac{\delta H}{\delta \mathbf{q}},$$

where $\delta H/\delta \mathbf{q}$ is the vector variational or Euler derivative of H with respect to $\mathbf{q}$. In addition, the bracket defined by

$$(3.3) \qquad [F, G] \equiv \left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta G}{\delta \mathbf{q}} \right\rangle,$$

where F and G are arbitrary allowable functionals of $\mathbf{q}$ (where the inner product is typically the $\mathbf{L}^2$ inner product) must satisfy the properties of self-commutation and skew symmetry, the distributive and associative properties, and the Jacobi identity given by, respectively,

$$(3.4) \qquad [F, F] = 0,$$

$$(3.5) \qquad [F, G] = -[G, F],$$

$$(3.6) \qquad [\alpha F + \beta G, Q] = \alpha[F, Q] + \beta[G, Q],$$

$$(3.7) \qquad [FG, Q] = F[G, Q] + [F, Q]G,$$

(3.8)
$$[F, [G, Q]] + [G, [Q, F]] + [Q, [F, G]] = 0,$$

where Q is an arbitrary allowable functional of $\mathbf{q}$ and α and β are arbitrary real numbers.

If the bracket (3.3) satisfies the five properties (3.4) through to (3.8), then we call the bracket a *Poisson bracket* and say that F *Poisson commutes with* G if $[F, G] = 0$. It is usually the case that the first four of these properties, i.e., (3.4) through to (3.7), are relatively easy to satisfy. The Jacobi identity puts a strong constraint on the structure of the $\mathbf{J}$ matrix and is usually the most difficult to verify.

Theorem 2. *The governing equations for fluid-filled hyperelastic tubes are Hamiltonian for the choice of*

$$(3.9) \qquad \mathbf{q} = (u, A)^T,$$

(3.10)
$$H(u, A) = \int_R \frac{Au^2}{2} + 2[W(a, (1 + e)\lambda) - W^\infty]_{a = A^{1/2}} \, dx,$$

$$(3.11) \qquad \mathbf{J} = -\begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix},$$

with $W^\infty \equiv W(a_\infty, 1 + e)$ *and where the Poisson bracket is given by*

(3.12)
$$[F, G] \equiv \int_R \begin{bmatrix} \dfrac{\delta F}{\delta u} & \dfrac{\delta F}{\delta A} \end{bmatrix} \mathbf{J} \begin{bmatrix} \dfrac{\delta G}{\delta u} & \dfrac{\delta G}{\delta A} \end{bmatrix}^T \, dx$$

$$= -\int_R \frac{\delta F}{\delta u} \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta A}\right) + \frac{\delta F}{\delta A} \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta u}\right) \, dx,$$

where the domain $R = \mathbf{R}$ or $R = \{x | 0 < x < L\}$ with the appropriate boundary conditions.

Proof. The proof requires several steps. First, we will show that the tube equations can be written in the form (3.2) for $\mathbf{q}$, H and $\mathbf{J}$ given by (3.9), (3.10) and (3.11), respectively. Second, we will show that the Hamiltonian is an invariant of the motion. We will then show that the bracket (3.12) satisfies the five properties (3.4) to (3.8).

The variational derivative $\delta H / \delta \mathbf{q} = (\delta H / \delta u, \delta H / \delta A)^T$ an be obtained from the first variation of H given by

$$
\begin{aligned}
(3.13) \quad \delta H &= \int_R Au\delta u + \left[\frac{u^2}{2} + 2\frac{\partial W}{\partial a}\frac{da}{dA} \right] \delta A + 2\frac{\partial W}{\partial \lambda}\frac{d\lambda}{da_x}\, \delta a_x \, dx \\
&= \int_R Au\delta u + \left\{ \frac{u^2}{2} + 2\frac{da}{dA}\left(\frac{\partial W}{\partial a} \right. \right. \\
&\qquad\qquad \left. \left. - \frac{\partial}{\partial x}\left[\frac{a_x}{(1 + a_x^2)^{1/2}}\frac{\partial W}{\partial \lambda} \right] \right) \right\} \delta A \, dx \\
&= \int_R Au\delta u + \left(\frac{u^2}{2} + p \right) dA \, dx,
\end{aligned}
$$

where we have integrated by parts once exploiting the boundary conditions and used (2.15) and (2.16). It therefore follows that

$$
(3.14) \qquad \frac{\delta H}{\delta u} = Au,
$$

$$
(3.15) \qquad \frac{\delta H}{\delta A} = \frac{u^2}{2} + p.
$$

Consequently,

$$
\begin{aligned}
(3.16) \quad \mathbf{J}\frac{\delta H}{\delta \mathbf{q}} &= - \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta u} & \frac{\delta H}{\delta A} \end{bmatrix}^T \\
&= - \begin{bmatrix} 0 & \partial_x \\ \partial_x & 0 \end{bmatrix} \begin{bmatrix} Au \\ u^2/2 + p \end{bmatrix} \\
&= \begin{bmatrix} -uu_x - p_x \\ -(Au)_x \end{bmatrix} = \begin{bmatrix} u \\ A \end{bmatrix}_t = \mathbf{q}_t.
\end{aligned}
$$

To show that the Hamiltonian is an invariant we proceed directly. We have

$$
\begin{aligned}
\frac{dH}{dt} &= \int_R (Au^2)_t + 2W_t \, dx \\
&= \int_R (Au^2)_t + 2\frac{\partial W}{\partial a}\frac{da}{dA}A_t + 2\frac{\partial W}{\partial \lambda}\frac{d\lambda}{da_x}a_{xt} \, dx \\
&= \int_R (Au^2)_t + \frac{1}{2}\left(\frac{\partial W}{\partial a} - \frac{\partial}{\partial x}\left[\frac{a_x}{(1+a_x^2)^{1/2}}\frac{\partial W}{\partial \lambda}\right]\right)A_t \, dx \\
&= \int_R (Au^2)_t + pA_t \, dx \\
&= \int_R (Au^2)_t - p(uA)_x \, dx \\
&= \int_R (Au^2)_t + uAp_x - (puA)_x \, dx \\
&= -\int_R \left(\frac{1}{2}u^3 A + puA\right)_x \, dx \\
&= 0,
\end{aligned}
$$

where we have integrated by parts as needed and used the energy equation

$$
(Au^2)_t + \left(\frac{1}{2}u^3 A\right)_x = -uAp_x,
$$

which follows from forming $Au(2.13) + (2.14)$.

All that remains to be shown is that the bracket (3.12) satisfies the five algebraic properties (3.4) through to (3.8). As it runs out, the demonstration of these properties requires integrating by parts. In order that the boundary terms vanish in the case that $R = \mathbf{R}$ it is necessary to restrict the class of allowable functionals to those functionals whose variational derivatives smoothly vanish at infinity or are *identically* constant. It will be shown in Section 3.2 that these later functionals belong to a special class called *Casimirs*.

For self-commutation we have

$$
\begin{aligned}
[F, F] &= -\int_R \frac{\delta F}{\delta u}\frac{\partial}{\partial x}\left(\frac{\delta F}{\delta A}\right) + \frac{\delta F}{\delta A}\frac{\partial}{\partial x}\left(\frac{\delta F}{\delta u}\right) \, dx \\
&= -\int_R \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta u}\frac{\delta F}{\delta A}\right) \, dx = 0,
\end{aligned}
$$

assuming $F = F(u, A)$ is an allowable functional.

To show that the bracket is skew-symmetric we proceed in a similar manner

$$[F, G] = -\int_R \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta G}{\delta A} \right) + \frac{\delta F}{\delta A} \frac{\partial}{\partial x} \left(\frac{\delta G}{\delta u} \right) dx$$

$$= -\int_R \frac{\partial}{\partial x} \left(\frac{\delta F}{\delta u} \frac{\delta G}{\delta A} + \frac{\delta G}{\delta u} \frac{\delta F}{\delta A} \right) dx$$

$$+ \int_R \frac{\delta G}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta F}{\delta A} \right) + \frac{\delta G}{\delta A} \frac{\partial}{\partial x} \left(\frac{\delta F}{\delta u} \right) dx$$

$$= \int_R \frac{\delta G}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta F}{\delta A} \right) + \frac{\delta G}{\delta A} \frac{\partial}{\partial x} \left(\frac{\delta F}{\delta u} \right) dx$$

$$= -[F, G],$$

if F and G are allowable functionals. Note that it is not necessary that the *matrix* $\mathbf{J}$ be skew-symmetric. Indeed, in this case $\mathbf{J}$ is a symmetric matrix.

The associative and distributive properties follow easily from the linearity of the variational derivative, i.e.,

$$[\alpha F + \beta G, Q] = -\int_R \frac{\delta(\alpha F + \beta G)}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta A} \right)$$

$$+ \frac{\delta(\alpha F + \beta G)}{\delta A} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta u} \right) dx$$

$$= -\alpha \int_R \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta A} \right) + \frac{\delta F}{\delta A} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta u} \right) dx$$

$$- \beta \int_R \frac{\delta G}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta A} \right) + \frac{\delta G}{\delta A} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta u} \right) dx$$

$$= \alpha[F, Q] + \beta[G, Q],$$

and

$$[FG, Q] = -\int_R \frac{\delta(FG)}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta A} \right) + \frac{\delta(FG)}{\delta A} \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta u} \right) dx$$

$$= -\int_R \left[F \frac{\delta G}{\delta u} + \frac{\delta F}{\delta u} G \right] \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta A} \right) dx$$

$$- \int_R \left[F \frac{\delta G}{\delta A} + \frac{\delta F}{\delta A} G \right] \frac{\partial}{\partial x} \left(\frac{\delta Q}{\delta u} \right) dx$$

$$= F[G, Q] + [F, Q]G.$$

All that remains to be shown is the Jacobi identity (3.8). It follows from the skew symmetry property and (3.3) that

$$(3.17) \quad [F,[G,Q]] + [G,[Q,F]] + [Q,[F,G]]$$
$$= -[[G,Q],F] - [[q,F],G] - [[F,G],Q]$$
$$= -\left\langle \frac{\delta}{\delta \mathbf{q}}[G,Q], \mathbf{J}\frac{\delta F}{\delta \mathbf{q}} \right\rangle$$
$$- \left\langle \frac{\delta}{\delta \mathbf{q}}[Q,F], \mathbf{J}\frac{\delta G}{\delta \mathbf{q}} \right\rangle$$
$$- \left\langle \frac{\delta}{\delta \mathbf{q}}[F,G], \mathbf{J}\frac{\delta Q}{\delta \mathbf{q}} \right\rangle$$
$$= -\left\langle \frac{\delta}{\delta \mathbf{q}}\left\langle \frac{\delta G}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta Q}{\delta \mathbf{q}} \right\rangle, \mathbf{J}\frac{\delta F}{\delta \mathbf{q}} \right\rangle$$
$$- \left\langle \frac{\delta}{\delta \mathbf{q}}\left\langle \frac{\delta Q}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta F}{\delta \mathbf{q}} \right\rangle, J\frac{\delta G}{\delta \mathbf{q}} \right\rangle$$
$$- \left\langle \frac{\delta}{\delta \mathbf{q}}\left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta G}{\delta \mathbf{q}} \right\rangle, \mathbf{J}\frac{\delta Q}{\delta \mathbf{q}} \right\rangle.$$

To proceed further we need to compute

$$\frac{\delta}{\delta \mathbf{q}}\left\langle \frac{\delta G}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta Q}{\delta \mathbf{q}} \right\rangle, \qquad \frac{\delta}{\delta \mathbf{q}}\left\langle \frac{\delta Q}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta F}{\delta \mathbf{q}} \right\rangle$$

and

$$\frac{\delta}{\delta \mathbf{q}}\left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta G}{\delta \mathbf{q}} \right\rangle.$$

We will compute the first of these and the others will follow in an obvious way. Using index notation with the summation convention, it follows that

$$\left\langle \frac{\delta G}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta Q}{\delta \mathbf{q}} \right\rangle = \int_R \frac{\delta G}{\delta q_i} J_{ij} \frac{\delta Q}{\delta q_j}\, dx,$$

so that the first variation can be written in the form

$$\delta \left\langle \frac{\delta G}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta Q}{\delta \mathbf{q}} \right\rangle = \int_R \frac{\delta^2 G}{\delta q_i \delta q_k}\delta q_k J_{ij}\frac{\delta Q}{\delta q_j} + \frac{\delta G}{\delta q_i}J_{ij}\left(\frac{\delta^2 Q}{\delta q_j \delta q_k}\delta q_k \right) dx$$
$$= \int_R \left[\frac{\delta^2 G}{\delta q_i \delta q_k}J_{ij}\frac{\delta Q}{\delta q_j} - \frac{\delta^2 Q}{\delta q_j \delta q_k}J_{ij}\frac{\delta G}{\delta q_i} \right]\delta q_k\, dx.$$

Consequently, we conclude that

$$\frac{\delta}{\delta q_k}\left\langle \frac{\delta G}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta Q}{\delta \mathbf{q}}\right\rangle = \frac{\delta^2 G}{\delta q_i \delta q_k}J_{ij}\frac{\delta Q}{\delta q_j} - \frac{\delta^2 Q}{\delta q_j \delta q_k}J_{ij}\frac{\delta G}{\delta q_i},$$

$$\frac{\delta}{\delta q_k}\left\langle \frac{\delta Q}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta F}{\delta \mathbf{q}}\right\rangle = \frac{\delta^2 Q}{\delta q_j \delta q_k}J_{ij}\frac{\delta F}{\delta q_i} - \frac{\delta^2 F}{\delta q_j \delta q_k}J_{ij}\frac{\delta Q}{\delta q_i},$$

$$\frac{\delta}{\delta q_k}\left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathbf{J}\frac{\delta G}{\delta \mathbf{q}}\right\rangle = \frac{\delta^2 F}{\delta q_j \delta q_k}J_{ij}\frac{\delta G}{\delta q_i} - \frac{\delta^2 G}{\delta q_i \delta q_k}J_{ij}\frac{\delta F}{\delta q_j},$$

where we have exploited the fact that $J_{ij} = J_{ji}$. If these expressions are substituted back into (3.17), it follows that

$$[F,[G,Q]] + [G,[Q,F]] + [Q,[F,G]]$$

$$= \int_R \frac{\delta^2 G}{\delta q_i \delta q_k}\left[\left(J_{km}\frac{\delta Q}{\delta q_m}\right)\left(J_{ij}\frac{\delta F}{\delta q_j}\right) - \left(J_{kj}\frac{\delta Q}{\delta q_j}\right)\left(J_{im}\frac{\delta F}{\delta q_m}\right)\right]dx$$

$$+ \int_R \frac{\delta^2 Q}{\delta q_j \delta q_k}\left[\left(J_{ij}\frac{\delta G}{\delta q_i}\right)\left(J_{km}\frac{\delta F}{\delta q_m}\right) - \left(J_{ik}\frac{\delta F}{\delta q_i}\right)\left(J_{jm}\frac{\delta G}{\delta q_m}\right)\right]dx$$

$$+ \int_R \frac{\delta^2 F}{\delta q_j \delta q_k}\left[\left(J_{ij}\frac{\delta Q}{\delta q_i}\right)\left(J_{km}\frac{\delta G}{\delta q_m}\right) - \left(J_{ki}\frac{\delta G}{\delta q_i}\right)\left(J_{mj}\frac{\delta Q}{\delta q_m}\right)\right]dx$$

$$= 0,$$

since the integrand in each integral is identically zero and where we have again used the fact that $J_{ij} = J_{ji}$. This completes the proof of the theorem. □

It is also possible to describe the dynamics in terms of the Poisson bracket. It is straightforward to verify that given an arbitrary functional $F = F(\mathbf{q})$, we have $F_t = [F, H]$. In particular, we may rewrite the tube equations in the form $\mathbf{q}_t = [\mathbf{q}, H]$, provided we interpret $\mathbf{q}$ in the Poisson bracket as the functional

$$\mathbf{q}(x,t) = \int_R \delta(\xi - x)\mathbf{q}(\xi, t)\, d\xi.$$

In our presentation of the noncanonical Hamiltonian formulation for the fluid-filled hyperelastic tube equations, we simply *guessed* the correct $\mathbf{J}$ operator and went on to show that it satisfied the requisite algebraic properties. The calculations required to verify these properties

while relatively straightforward in this case are, in general, somewhat
lengthy and tedious. A much better procedure, particularly for phys-
ically relevant applications which are sufficiently different from previ-
ously worked out Hamiltonian formulations, is to derive the Poisson
bracket written in terms of the desired variables from a systematic re-
duction of the *canonical* Poisson bracket written in terms of the canoni-
cal Lagrangian positions and momenta. If the required Poisson bracket
is derived in this manner, then (3.3) can be inverted to determine **J**.

3.2. *Impulse invariant and Casimir functional.* There are two classes
of invariants in Hamiltonian dynamics. The first are those which can
be directly identified as a consequence of Noether's theorem (e.g.,
Courant and Hilbert [9]). These are the invariants corresponding
to a symmetry in the Hamiltonian. For example, invariance of the
Hamiltonian to translations in t implies energy conservation or the
invariance of the Hamiltonian to translations in x implies conservation
of linear momentum, etc.

The other class of invariants that are of interest are those associated
with the degeneracy of the Hamiltonian formulation. These are the
Casimirs (Holm et al., [15]). They do *not* correspond to any physical
symmetry. Formally, Casimirs are those functionals which Poisson
commute with all other functionals.

Definition 3.   A Casimir functional $C = C(\mathbf{q})$ is a functional
satisfying

$$(3.18) \qquad [F, C] = 0, \quad \forall F = F(\mathbf{q}).$$

Substitution of (3.3) into (3.18) implies that the complete family of
Casimirs is described by the solutions to

$$\left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathbf{J} \frac{\delta C}{\delta \mathbf{q}} \right\rangle = 0,$$

and since $\delta F/\delta \mathbf{q}$ is arbitrary this is equivalent to

$$(3.19) \qquad \mathbf{J} \frac{\delta C}{\delta \mathbf{q}} = \mathbf{0}.$$

If $\mathbf{J}^{-1}$ exists, then it follows from (3.19) that the Casimirs are trivial (i.e., constant) and we say that the Hamiltonian formulation is *canonical*. However, if $\mathbf{J}$ is noninvertible, then there will exist nontrivial Casimirs and we say that the Hamiltonian formulation is *noncanonical*. Casimirs are clearly invariants since by definition $[H, C] = 0 \Leftrightarrow C_t = 0$.

Theorem 4. *The Hamiltonian formulation* (3.9), (3.10) *and* (3.11) *is noncanonical and the Casimirs are given by*

$$(3.20) \qquad C(u, A) = \int_R \alpha u + \beta (A - 1) \, dx,$$

where α and β are arbitrary real numbers.

Proof. Substitution of (3.9) and (3.10) into (3.19) implies that

$$(3.21) \qquad \left(\frac{\delta C}{\delta u} \right)_x = 0 \Longrightarrow \frac{\delta C}{\delta u} = \alpha,$$

$$(3.22) \qquad \left(\frac{\delta C}{\delta A} \right)_x = 0 \Longrightarrow \frac{\delta C}{\delta A} = \beta.$$

The required invariance of the Casimirs precludes the possibility that the integration constants are functions of time. Consequently, the relations (3.21) and (3.22) imply (3.20). □

The other invariant of interest here is the linear momentum or impulse, denoted by $I = I(\mathbf{q})$, associated with the invariance of the Hamiltonian to translations in x. the impulse [6] is an invariant functional satisfying

$$(3.23) \qquad \mathbf{J} \frac{\delta I}{\delta \mathbf{q}} = -\mathbf{q}_x.$$

Theorem 5. *The impulse for the tube equations is given by*

$$(3.24) \qquad I(u, A) = \int_R (A - 1) u \, dx.$$

Proof. It follows from (3.24) that $\delta I/\delta u = A - 1$, $\delta I/\delta A = u$. It therefore follows that

$$\mathbf{J}\frac{\delta I}{\delta \mathbf{q}} = \begin{bmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{bmatrix} \begin{bmatrix} A - 1 \\ u \end{bmatrix} = \begin{bmatrix} -u_x \\ -A_x \end{bmatrix} = -\mathbf{q}_x.$$

The invariance of I can be shown directly

$$\begin{aligned}
I_t &= \int_R [(A-1)u]_t \, dx \\
&= \int_R \frac{1}{2}[(1-A)u^2]_x + (1-A)p_x \, dx \\
&= \int_R pA_x \, dx \\
&= 2\int_R \left\{ \frac{W_a}{a} - \frac{1}{a}\frac{\partial}{\partial x}\left[\frac{a_x}{(1+a_x^2)^{1/2}}W_\lambda \right] \right\} aa_x \, dx \\
&= 2\int_R \left[a_x W_a + \frac{a_x a_{xx}}{(1+a_x^2)^{1/2}}W_\lambda \right] dx \\
&= 2\int_R W_x \, dx \\
&= 2\int_R (W - W^\infty)_x \, dx \\
&= 0. \qquad \square
\end{aligned}$$

4. Steadily-travelling solutions and a variational principle.

4.1. *Steadily-travelling solutions.* In what follows we will restrict attention, for the most part, to steadily-travelling solutions which satisfy the boundary conditions (2.19) to (2.21) or (2.22) to (2.24). Substitution of

(4.1) $$u(x,t) = u_s(\xi),$$

(4.2) $$a(x,t) = a_s(\xi),$$

(4.3) $$p(x,t) = p_s(\xi),$$

where $\xi \equiv x - ct$ into the governing equations (2.13) and (2.14) implies

(4.4) $$(u_s - c)^2/2 + p_2 = B,$$

(4.5) $$A_s(u_s - c) = M,$$

where B and M are, as yet, undetermined constants of integration. (Note that a subscript with respect to s does not denote differentiation.) Equations (4.4) and (4.5) can be combined together to give

(4.6) $$p_s + M^2/(2a_s^4) = B.$$

If we demand that the uniformly pre-stressed state

(4.7) $$u_2 = 0, \qquad a_2 = 1, \qquad p_s = W_a^\infty,$$

is *always* a solution, it follows from (4.4) and (4.5) that

(4.8) $$B = W_a^\infty + c^2/2,$$
(4.9) $$M = -c,$$

and (4.6) can be written in the form

(4.10) $$\frac{\partial}{\partial\xi}\left[\frac{a_{s\xi}W_\lambda}{(1 + a_{s_\xi}^2)^{1/2}}\right] + W_a^\infty a_s - W_a^s + \frac{c^2}{2}\left(a_s - \frac{1}{a_s^3}\right) = 0,$$

where

$$W_a^s \equiv [W_a]_{(a,\lambda)=(a_s,\lambda_s)}, \qquad \lambda_s \equiv (1 + a_{s_\xi}^2)^{1/2}.$$

We defer a detailed proof of the existence of fully nonlinear periodic and solitary solutions to (4.10) until a future paper.

We can recover the infinitesimally-small periodic solutions by introducing

(4.11) $$a_s = 1 + \varepsilon\varphi(\xi), \qquad 0 < \varepsilon \ll 1.$$

Substitution of (4.11) into (4.10), neglecting terms of $O(\varepsilon^2)$ and higher yields the leading order problem

(4.12) $$\varphi_{\xi\xi} + \frac{(2c^2 + W_a^\infty - W_{aa}^\infty)}{W_\lambda^\infty}\varphi = 0.$$

Assuming a periodic solution of the form

$$\varphi(\xi) = \varphi_{amp}\exp(ik\xi) + c.c.,$$

gives the dispersion relation

$$(4.13) \qquad 2c^2 = W_{aa}^\infty - W_a^\infty + k^2 W_\lambda^\infty, \qquad n = 0, 1, 2, \dots,$$

where we define

$$(4.14) \qquad W_{\nu_1 \cdots \nu_n}^\infty \equiv \left[\frac{\partial^n W}{\partial \nu_1 \cdots \partial \nu_n} \right]_{(a,\lambda)=(1,1)}.$$

These solutions satisfy the periodic boundary conditions (2.22) to (2.24) for $L \equiv 2\pi/k$. Note that it follows from (4.13) that c will always be real since for a hyperelastic material it is known that $W_{aa}^\infty - W_a^\infty > 0$ and $W_\lambda^\infty > 0$ [21]. The dispersion relation (4.13) corresponds to the known dispersion relation for linear dispersive waves in a tethered hyperelastic tube with radial variations ignored [30].

Small-but-finite amplitude solitary and cnoidal wave solutions can be recovered with the weakly-nonlinear long wavelength scaling

$$(4.15) \qquad a_s = 1 + \varepsilon\varphi(\chi), \qquad \chi = \varepsilon^{1/2}\xi, \qquad 0 < \varepsilon \ll 1.$$

Substitution of (4.15) into (4.10), neglecting terms of $O(\varepsilon^3)$ and higher, leads to

$$(4.16) \quad (W_a^\infty - W_{aa}^\infty + 2c^2)\varphi + \varepsilon W_\lambda^\infty \varphi_{\chi\chi} \\ - \varepsilon(W_{aaa}^\infty/2 + 3c^2)\varphi^2 + O(\varepsilon^2) = 0.$$

If the expansion

$$(4.17) \qquad \varphi(\chi) \simeq \varphi^{(0)}(\chi) + \varepsilon\varphi^{(1)}(\chi) + O(\varepsilon^2),$$

$$(4.18) \qquad c \simeq c^{(0)} + \varepsilon c^{(1)} + O(\varepsilon^2),$$

is inserted into (4.16), the $O(1)$ and $O(\varepsilon)$ problems are, respectively,

$$(4.19) \qquad\qquad [W_a^\infty - W_{aa}^\infty + 2(c^{(0)})^2]\varphi^{(0)} = 0,$$

$$(4.20)$$
$$W_\lambda^\infty \varphi_{\chi\chi}^{(0)} + 4c^{(0)}c^{(1)}\varphi^{(0)} - [W_{aaa}^\infty/2 + 3(c^{(0)})^2](\varphi^{(0)})^2 = 0.$$

From (4.19) we conclude that the leading order translation velocity is determined by

$$(4.21) \qquad\qquad (c^{(0)})^2 = (W_{aa}^\infty - W_a^\infty)/2 > 0,$$

which is just the classical Korteweg-Moens relation for nondispersive waves in a tethered hyperelastic fluid-filled tube (e.g., [10]).

If the transformation

$$(4.22) \qquad \varphi^{(0)}(\xi) = \frac{12|c^{(0)}|\Phi(\eta)}{[W_{aaa}^\infty/2 + 3(c^{(0)})^2]}, \qquad \eta = \left(\frac{4|c^{(0)}|}{W_\lambda^\infty}\right)^{1/2} \chi,$$

is introduced into (4.20), it follows that

$$(4.23) \qquad \Phi_{\eta\eta} = \sigma\Phi + 3\Phi^2,$$

where $\sigma \equiv -c^{(1)}\text{sgn}(c^{(0)})$. If (4.23) is multiplied through by Φ_η and the result integrated, it follows that

$$(4.24) \qquad (\Phi_\eta)^2/2 = \Gamma(\Phi) \equiv \Phi^3 + \frac{\sigma}{2}\Phi^2 + \Phi_\infty,$$

where Φ_∞ is a constant of integration.

For solitary wave solutions satisfying (2.19) to (2.21), it follows that $\Phi_\infty = 0$ and consequently that $(\Phi_\eta)^2 = \Phi^2(2\Phi + \sigma)$, which has the soliton solution (Drazin and Johnson [12])

$$(4.25) \qquad \Phi(\eta) = -\frac{\sigma}{2}\text{sech}^2\left[\frac{\sqrt{\sigma}}{2}(\eta - \eta_0)\right],$$

where η_0 is an arbitrary phase shift parameter. Note that $\sigma > 0$ for this solution to exist which implies that the sign of $c^{(1)}$ is necessarily opposite to the sign of $c^{(0)}$. The soliton (4.25) travels, therefore, with a velocity which lies outside the range of the translation velocities associated with the linear dispersive waves given by (4.13).

If $\Phi_\infty \neq 0$, there are bounded solutions *only if* the roots of $\Gamma(\Phi)$ are all real [12]. If the three real roots of $\Gamma(\Phi)$ are denoted ϕ_1, ϕ_2 and ϕ_3, respectively, with the ordering $\phi_3 < \phi_2 < \phi_1$, then the solution to (4.24) can be written using the $cn[*_1 \mid *_2]$ Jacobi elliptic function in the form

$$(4.26) \qquad \Phi(\eta) = \phi_2 + (\phi_3 - \phi_2)cn^2\left[\sqrt{\frac{(\phi_1 - \phi_3)}{2}}(\eta - \eta_0) \mid m\right],$$

where

$$m \equiv \frac{\phi_2 - \phi_3}{\phi_1 - \phi_3} > 0.$$

The solutions described by (4.26) correspond to periodic cnoidal waves which oscillate between the levels ϕ_2 and ϕ_3. The wavelength of these waves is given by

$$(4.27) \qquad L = 2K(m)\sqrt{\frac{2}{(\phi_1 - \phi_3)}},$$

where $K(m)$ is the complete elliptic function of the first kind given by

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - m\sin^2(\theta))}}.$$

If one writes $\Gamma(\Phi) = (\Phi - \phi_1)(\Phi - \phi_2)(\Phi - \phi_3)$, then comparing this expression with (4.24) leads to

$$(4.28) \qquad c^{(1)} = 2(\phi_1 + \phi_2 + \phi_3)\mathrm{sgn}\,(c^{(0)}),$$

which determines the translation velocity as a function of the roots of $\Gamma(\Phi)$.

There is also a special *soliton on a background* solution. Assuming that $\sigma = -3(2\Phi_\infty)^{1/3} \Rightarrow c^{(1)} = 3(2\Phi_\infty)^{1/3}\mathrm{sgn}\,(c^{(0)})$ where $\Phi_\infty > 0$, it follows that the roots of $\Gamma(\Phi)$ are given by $\{-(\Phi_\infty/4)^{1/3}, (2\Phi_\infty)^{1/3}, (2\Phi_\infty)^{1/3}\}$. These roots imply that $m = 1$, which in turn implies that (4.26) takes the form

$$(4.29) \quad \Phi(\eta) = (2\Phi_\infty)^{1/3} - \left(\frac{27\Phi_\infty}{4}\right)^{1/3} \mathrm{sech}^2\left[\left(\frac{27\Phi_\infty}{32}\right)^{1/6}(\eta - \eta_0)\right].$$

These solutions correspond to a solitary wave which decays to a uniformly stressed state at infinity which differs from the tethered prestressed state (2.19) and (2.20). The soliton solution (4.29) is singular in the sense that it does not correspond to the branch of soliton solutions described by (4.25). Observe that in the limit $\Phi_\infty \to 0$ the roots $\{-(\Phi_\infty/4)^{1/3}, (2\Phi_\infty)^{1/3}, (2\Phi_\infty)^{1/3}\} \to \{0, 0, 0\}$ and thus $\Phi \to 0$

for the solution (4.29). In the case where $\Phi_\infty < 0$ it follows that $\phi_2 = \phi_3 \Rightarrow m = 0$ for these solutions and thus (4.26) implies that $\Phi(\eta)$ is identically constant.

4.2. Variational principle. In classical soliton theory (e.g., [5]), it can be shown that the steadily-travelling solutions satisfy the first-order conditions for an extremal of the Hamiltonian constrained by the impulse. It will be shown here that this principle holds provided we introduce an appropriate *Casimir* functional as a further constraint. Similar constraints are known to be needed in other noncanonical Hamiltonian systems (e.g., [15]).

Consider the constrained Hamiltonian given by

$$(4.30) \qquad \mathcal{H}(u, A) = H(u, A) - cM(u, A) + C(u, A),$$

where H, M and C are given by (3.10), (3.24) and (3.20), respectively. Substitution into (4.30) yields

$$(4.31) \quad \mathcal{H} = \int_R \{ Au^2/2 + 2[W(a, (1+e)\lambda) - W^\infty]_{a=A^{1/2}} \\ + \alpha u + \beta(A-1) - c(A-1)u \} \, dx.$$

Clearly, $\mathcal{H}(u, A)$ is an invariant of the motion since H, M and C are individually invariant.

The first variation of $\mathcal{H}$ is given by

$$\delta \mathcal{H}(u, A) = \int_R [Au + \alpha - c(A-1)] \, du + [u^2/2 + p + \beta - cu] \, dA \, dx,$$

implying that

$$(4.32) \qquad \delta \mathcal{H}/\delta u = A(u-c) + c + \alpha,$$
$$(4.33) \qquad \delta \mathcal{H}/\delta A = (u-c)^2/2 + p - c^2/2 + \beta.$$

Consequently, we see that the steadily-travelling solutions (4.4) and (4.5) with (4.8) and (4.9) (or equivalently (4.10)) imply $\delta \mathcal{H}(u_s, A_s)/\delta \mathbf{u} = 0$, provided

$$(4.34) \qquad\qquad \alpha = 0,$$
$$(4.35) \qquad\qquad \beta = -W_a^\infty.$$

Consequently, we have proved

Theorem 6. *The steadily-travelling solutions $a_s(x)$, $A_s(x)$, $u_s(x)$ and $p_s(x)$ given by*

$$(u_s - c)^2/2 + p_2 = c^2/2 + W_a^\infty,$$
$$A_s(c - u_s) = c,$$

subject to the boundary conditions (2.19) *to* (2.21) *or* (2.22) *to* (2.24) *satisfy the variational principle*

$$\delta\mathcal{H}(u_s, A_s) = 0,$$

for the constrained Hamiltonian

$$(4.36) \quad \mathcal{H}(u, A) = \int_R \left\{ \frac{Au^2}{2} + 2[W(a, (1 + e)\lambda) - W^\infty]_{a=A^{1/2}} - (W_a^\infty + uc)(A - 1) \right\} dx.$$

5. Dynamical characteristics.

5.1. *Linear stability.* The governing equations (2.13) and (2.14) written in a frame of reference moving with velocity c can be written in the form

$$(5.1) \qquad\qquad u_t + [(u - c)^2/2 + p]_x = 0,$$
$$(5.2) \qquad\qquad (a^2)_t + [(u - c)a^2]_x = 0,$$

where p remains determined by (2.15). With respect to this frame, the steadily-travelling solutions determined (4.4) and (4.5) appear as *steady* solutions to (5.1) and (5.2).

The linear stability equations are obtained by assuming solutions of the form

$$(5.3) \qquad\qquad u(x, t) = u_s(x) + \tilde{u}(x, t),$$
$$(5.4) \qquad\qquad p(x, t) = p_s(x) + \tilde{p}(x, t),$$
$$(5.5) \qquad\qquad a(x, t) = a_s(x) + \tilde{a}(x, t),$$

where the terms with the tilde will be called the perturbation terms. Substitution of these expressions into (5.1), (5.2) and (2.15) leads to

(5.6) $u_t + [(u_s - c)u + p]_x = 0,$

(5.7) $a_s a_t + [a_s^2 u/2 + a a_s (u_s - c)]_x,$

$$(5.8) \quad \begin{aligned} p = &-\frac{p_s a}{a_s} + \frac{1}{a_s}\left[W_{aa}^s a + \frac{W_{a\lambda}^s a_{sx} a_x}{(1 + a_{sx}^2)^{1/2}} \right] \\ &- \frac{1}{a^s}\frac{\partial}{\partial x}\left[\frac{W_\lambda^s a_x}{(1 + a_{s_x}^2)^{3/2}} + \frac{W_{a\lambda}^s a_{s_x} a}{(1 + a_{s_x}^2)^{1/2}} + \frac{W_{\lambda\lambda}^s a_{s_x}^2 a}{(1 + a_{s_x}^2)} \right], \end{aligned}$$

where we have neglected all quadratic and higher order perturbation terms and dropped the tilde notation, and where we define

$$W_{\nu_1 \cdots \nu_n}^s \equiv \left[\frac{\partial^n W}{\partial\nu_1 \cdots \partial\nu_n} \right]_{(a,\lambda)=(a_s, \lambda_s)}.$$

Clearly, the linear stability problem is, in general, quite complicated. Nevertheless, it is possible to obtain relatively simple linear stability criterion for the steadily-travelling solutions. It is well known (e.g., Holm et al., [15]) that the second variation associated with the constrained Hamiltonian $\mathcal{H}$ (for which the variational principle holds) evaluated at the steadily-travelling solution $a_s(x)$ is necessarily an invariant of the linear stability equations. If conditions can be found that ensure that the second variation is definite for all suitable perturbations, then the steadily-travelling solution $a_s(x)$ is said to be *formally stable* [15] and it will be possible to establish the *linear* stability in the sense of Liapunov for $a_s(x)$; thereby eliminating both algebraic *and* normal mode instabilities.

However, because the underlying phase space is an infinite dimensional Hilbert space, the lack of compactness means that the definiteness of the second variation is not sufficient to prove the nonlinear stability in the sense of Liapunov. The nonlinear stability theorem given here requires additional global convexity assumptions to make sure that the steadily-travelling solution is a strict local extremum of the constrained Hamiltonian. The approach we take here is similar to

the derivation of the stability conditions for solutions to the shallow-water equations (e.g., Ripa [23]).

The second variation of $\mathcal{H}$ can be written in the form
(5.9)
$$
\begin{aligned}
\delta^2 \mathcal{H}\,(u, A) &= \int_R [A(u{-}c) + c]\delta^2 u + \left[\frac{(u-c)^2}{2} + p - W_a^\infty - \frac{c^2}{2}\right]\delta^2 A \\
&\quad + A(\delta u)^2 + 2(u - c)\delta u \delta A + \delta p \delta A\, dx \\
&= \int_R [A(u - c) + c]\delta^2 u + \left[\frac{(u - c)^2}{2} + p - W_a^\infty - \frac{c^2}{2}\right]\delta^2 A \\
&\quad + A\left[\delta u + \frac{(u{-}c)}{A}\delta A\right]^2 - 4(u{-}c)^2(\delta a)^2 + 2a\delta p \delta a\, dx.
\end{aligned}
$$

However, from (2.15) it follows that

(5.10)
$$
\begin{aligned}
\delta p &= -\frac{p\delta a}{a} + \frac{1}{a}\left[W_{aa}\delta a + \frac{W_{a\lambda}a_x\delta a_x}{(1 + a_x^2)^{1/2}}\right] \\
&\quad - \frac{1}{a}\frac{\partial}{\partial x}\left[\frac{W_\lambda \delta a_x}{(1 + a_x^2)^{3/2}} + \frac{W_{a\lambda}a_x\delta a}{(1 + a_x^2)^{1/2}} + \frac{W_{\lambda\lambda}a_x^2\delta a_x}{(1 + a_x^2)}\right].
\end{aligned}
$$

Substitution of (5.10) into (5.9) leads to, after integrating by parts where necessary,

(5.11)
$$
\begin{aligned}
\delta^2 \mathcal{H}\,(u, A) &= \int_R [A(u{-}c) + c]\delta^2 u + \left[\frac{(u-c)^2}{2} + p - W_a^\infty - \frac{c^2}{2}\right]\delta^2 A \\
&\quad + A\left[\delta u + \frac{2(u - c)}{a}\delta a\right]^2 \\
&\quad + 2\left\{W_{aa} - p - 2(u - c)^2 - \frac{\partial}{\partial x}\left[\frac{W_{a\lambda}a_x}{(1 + a_x^2)^{1/2}}\right]\right\}(\delta a)^2 \\
&\quad + 2\left\{\frac{W_\lambda}{(1 + a_x^2)^{3/2}} + \frac{W_{\lambda\lambda}a_x^2}{(1 + a_x^2)}\right\}(\delta a_x)^2\, dx.
\end{aligned}
$$

Therefore, we have

(5.12)

$$
\delta^2 \mathcal{H}\,(u_s, A_s) = \int_R A_s \left[\delta u + \frac{2(u_s - c)}{a_s} \delta a \right]^2
$$
$$
+ 2 \left\{ W^s_{aa} - p_s - 2(u_s - c)^2 - \frac{\partial}{\partial x} \left[\frac{W^s_{a\lambda} a_{s_x}}{(1 + a^2_{s_x})^{1/2}} \right] \right\} (\delta a)^2
$$
$$
+ 2 \left\{ \frac{W^s_\lambda}{(1 + a^2_{s_x})^{3/2}} + \frac{W^s_{\lambda\lambda} a^2_{s_x}}{(1 + a^2_{s_x})} \right\} (\delta a_x)^2 \, dx
$$
$$
= \int_R A_s \left[\delta u + \frac{2(u_s - c)}{a_s} \delta a \right]^2
$$
$$
+ 2 \left\{ W^s_{aa} - W^\infty_a - \frac{c^2}{2} - \frac{3c^2}{2a^4_s} - \frac{\partial}{\partial x} \left[\frac{W^s_{a\lambda} a_{s_x}}{(1 + a^2_{s_x})^{1/2}} \right] \right\} (\delta a)^2
$$
$$
+ 2 \left\{ \frac{W^s_\lambda}{(1 + a^2_{s_x})^{3/2}} + \frac{W^s_{\lambda\lambda} a^2_{s_x}}{(1 + a^2_{s_x})} \right\} (\delta a_x)^2 \, dx,
$$

where (4.4), (4.6), (4.8) and (4.9) has been used and where $a_{s_x} \equiv \partial a_s(x)/\partial x$. It is (tediously) straightforward to show that $\delta^2 \mathcal{H}\,(u_s, A_s)$ is an invariant of the linear stability equations (5.6), (5.7) and (5.8) with (u, a) replaced by $(\delta u, \delta a)$.

It is not possible to obtain general conditions which imply $\delta^2 \mathcal{H}\,(u_s, A_s)$ is negative definite for all perturbations. However, if

(5.13)
$$
W^\infty_a + \frac{c^2}{2} < \inf_R \left\{ W^s_{aa} - \frac{3c^2}{2a^4_s} - \frac{\partial}{\partial x} \left[\frac{W^s_{a\lambda} a_{s_x}}{(1 + a^2_{s_x})^{1/2}} \right] \right\},
$$

(5.14)
$$
0 < \inf_R \left\{ \frac{W^s_\lambda}{(1 + a^2_{s_x})^{1/2}} + W^s_{\lambda\lambda} a^2_{s_x} \right\},
$$

both hold, then $\delta^2 \mathcal{H}\,(u_s, A_s) > 0$. If (5.13) and (5.14) hold it is possible to establish the linear stability in the sense of Liapunov for the steadily-travelling solution $a_s(x)$. One thing to note about (5.13) is that we are implicitly assuming that a_s is never zero, that is, we only consider steadily-travelling solutions that do not collapse.

Theorem 7. *Suppose the steadily-travelling solution* $a = a_s(x) > 0$ *satisfies* (5.13) *and* (5.14). *Then it is linearly stable in the sense of*

Liapunov with respect to the Sobolev-like perturbation norm

$$\|\delta \mathbf{u}\|^2 \equiv \int_R (\delta u)^2 + (\delta a)^2 + (\delta a_x)^2 \, dx.$$

Proof. Clearly, the conditions (5.13) and (5.14) are sufficient to establish the positive definiteness of $\delta^2 \mathcal{H}(u_s, A_s)$. All that remains is to establish the following a priori estimate. We have
(5.15)

$$\begin{aligned}
\|\delta \mathbf{u}\|^2 &= \int_R \left(\delta u + \frac{2(u_s - c)}{a_s} \delta a - \frac{2(u_s - c)}{a_s} \delta a \right)^2 + (\delta a)^2 + (\delta a_x)^2 \, dx \\
&\le \int_R 2 \left(\delta u + \frac{2(u_s - c)}{a_s} \delta a \right)^2 + \left[1 + 8 \left(\frac{u_s - c}{a_s} \right)^2 \right] (\delta a)^2 + (\delta a_x)^2 \, dx \\
&\le \Gamma_1 \int_R \left(\delta u + \frac{2(u_s - c)}{a_s} \delta a \right)^2 + (\delta a)^2 + (\delta a_x)^2 \, dx \\
&\le \Gamma_2 \delta^2 \mathcal{H}(u_s, A_s),
\end{aligned}$$

where

$$\Gamma_2 \equiv \Gamma_1 / \Gamma_3 > 0,$$

$$\Gamma_1 \equiv \max \left\{ 2, \sup_R \left[1 + 8 \left(\frac{u_s - c}{a_s} \right)^2 \right] \right\} > 0,$$

$$\Gamma_3 \equiv \min \{ \inf_R [A_s], 2\Gamma_4, 2\Gamma_5 \} > 0,$$

$$\Gamma_4 \equiv \inf_R \left\{ W_{aa}^s - W_a^\infty - \frac{c^2}{2} - \frac{3c^2}{2a_s^4} - \frac{\partial}{\partial x} \left[\frac{W_{a\lambda}^s a_{s_x}}{(1 + a_{s_x}^2)^{1/2}} \right] \right\} > 0,$$

$$\Gamma_5 \equiv \inf_R \left\{ \frac{W_\lambda^s}{(1 + a_{s_x}^2)^{3/2}} + \frac{W_{\lambda\lambda}^s a_{s_x}^2}{(1 + a_{s_x}^2)} \right\} > 0.$$

However, exploiting the invariance of $\delta^2 \mathcal{H}(u_s, A_s)$ we find
(5.16)

$$\begin{aligned}
\|\delta \mathbf{u}\|^2 &\le \Gamma_2 \delta^2 \mathcal{H}(u_s, A_s) = \Gamma_2 [\delta^2 \mathcal{H}(u_s, A_s)]_{t=0} \\
&\le \Gamma_6 \int_R \left(\delta u_0 + \frac{2(u_s - c)}{a_s} \delta a_0 \right)^2 + (\delta a_0)^2 + (\delta a_{0_x})^2 \, dx \\
&\le \Gamma_6 \int_R 2(\delta u_0)^2 + \left[1 + 8 \left(\frac{u_s - c}{a_s} \right)^2 \right] (\delta a_0)^2 + (\delta a_{0_x})^2 \, dx \\
&\le \Gamma_7 \|\delta \mathbf{u}_0\|^2,
\end{aligned}$$

where

$$\Gamma_7 \equiv \Gamma_6 \max \left\{ 2, \sup_R \left[1 + 8 \left(\frac{u_s - c}{a_s} \right)^2 \right] \right\} > 0$$

$$\Gamma_6 \equiv \Gamma_2 \max \{ \sup_R [A_s], 2\Gamma_8, 2\Gamma_9 \} > 0,$$

$$\Gamma_8 \equiv \sup_R \left\{ W_{aa}^s - W_a^\infty - \frac{c^2}{2} - \frac{3c^2}{2a_s^4} - \frac{\partial}{\partial x} \left[\frac{W_{a\lambda}^s a_{s_x}}{(1 + a_{s_x}^2)^{1/2}} \right] \right\} > 0,$$

$$\Gamma_9 \equiv \sup_R \left\{ \frac{W_\lambda^s}{(1 + a_{s_x}^2)^{3/2}} + \frac{W_{\lambda\lambda}^s a_{s_x}^2}{(1 + a_{s_x}^2)} \right\} > 0,$$

and where $(\delta a_0, \delta u_0) \equiv (\delta a, \delta u)_{t=0}$.

Therefore, for every $\varepsilon > 0$

$$||\delta \mathbf{u}_0|| < \varepsilon (\Gamma_7)^{-1/2} \implies ||\delta \mathbf{u}|| < \varepsilon,$$

for all $t > 0$. Thus the linear stability in the sense of Liapunov has been established for the steadily-travelling solutions provided (5.13) and (5.14) hold. □

5.2. *Nonlinear stability.* In this subsection we give conditions that can prove the nonlinear stability in the sense of Liapunov for a steadily-travelling solution. A cautionary note should be made here. Our demonstration will establish nonlinear stability with respect to the finite-amplitude perturbation Sobolev-like norm

$$||\mathbf{u}||^2 \equiv \int_R u^2 + a^2 + a_x^2 \, dx,$$

where $\mathbf{u} = (u, a)$ assuming certain convexity hypotheses on the strain energy function. The required assumptions are probably far too stringent for any practical situation. Of more interest would be a proof of the nonlinear stability of the steadily-travelling solutions with respect to form (i.e., with translations in x factored out) as argued by Benjamin [5] for the KdV soliton. However we, as yet, have been unable to establish stability with respect to form for steadily-travelling solutions to the general tube equations (if it exists). Because of this point, it is likely the case that of the two stability results presented in this paper, the linear stability result is the more physically relevant.

Consider the functional $\mathcal{N}(u,a)$ defined by

$$(5.17) \quad \mathcal{N}(u,a) = \mathcal{H}(u_s + u, (a_s + a)^2) - \mathcal{H}(u_s, a_s^2) - \delta\mathcal{H}(u_s, a_s^2),$$

where $(\delta u, \delta A)$ in $\delta\mathcal{H}(u_s, a_s^2)$ is replaced by $(u, 2a_s a)$. In this context (u,a) are finite-amplitude perturbations and $u_T \equiv u_s + u$ and $a_T \equiv a_s + a$ with $A_T \equiv (a_s + a)^2$ are solutions to (2.13), (2.14) and (2.15). The functional $\mathcal{N}(u,a)$ is an invariant of the nonlinear system (2.13) and (2.14) since the first two terms on the right-hand side of (5.17) are individually invariant and the third term is zero.

$\mathcal{N}(u,a)$ can be explicitly written out as, after a little algebra,

$$(5.18) \quad \mathcal{N}(u,a) = \int_R \frac{(a+a_s)^2}{2}\left\{u + \frac{(u_s - c)[(a + a_s)^2 - a_s^2]}{(a+a_s)^2}\right\}^2$$
$$- \frac{(u_s - c)^2[(a+a_s)^2 - a_s^2]^2}{2(a+a_s)^2} + \left(\frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty\right)a^2$$
$$+ 2[\widetilde{W} - W^s - W_a^s a - W_{a_x}^s a_x]\,dx,$$

where

$$\widetilde{W} \equiv W(a + a_s, (1+e)(1 + (a+a_s)_x^2)^{1/2}),$$
$$W^s \equiv W(a_s, (1+e)(1 + a_{s_x}^2)^{1/2}),$$
$$W_{a_x}^s \equiv \left[\frac{\partial W}{\partial a_x}\right]_{(a,\lambda)=(a_s,\lambda_s)}.$$

If $\mathcal{N}(u,a)$ is Taylor expanded about $(u,a) = (0,0)$, it is straightforward to check that

$$\mathcal{N}(u,a) \simeq \delta^2\mathcal{H}(u_s, A_s)/2 + h.o.t.,$$

with $(\delta u, \delta a)$ in $\delta^2\mathcal{H}(u_s, A_s)$ is replaced by (u,a).

Nonlinear will be established if we can find positive constants γ_1 and γ_2 satisfying

$$\gamma_1\|(u,a)\|^2 \le \mathcal{N}(u,a) \le \gamma_2\|(u,a)\|^2.$$

We begin with establishing the lower estimate as follows

$$(5.19)$$
$$\|(u,a)\|^2 = \int_R \left\{u + \frac{(u_s - c)[(a + a_s)^2 - a_s^2]}{(a+a_s)^2}\right.$$

$$- \frac{(u_s - c)[(a + a_s)^2 - a_s^2]}{(a + a_s)^2} \bigg\}^2 + a^2 + a_x^2 \, dx$$

$$\leq \int_R 2 \bigg\{ u + \frac{(u_s - c)[(a + a_s)^2 - a_s^2]}{(a + a_s)^2} \bigg\}^2$$

$$+ \frac{2(u_s - c)^2[(a + a_s)^2 - a_s^2]^2}{(a + a_s)^4} + a^2 + a_x^2 \, dx$$

$$\leq \int_R 2 \bigg\{ u + \frac{(u_s - c)[(a + a_s)^2 - a_s^2]}{(a + a_s)^2} \bigg\}^2$$

$$+ \bigg\{ 1 + 2 \frac{(u_s - c)^2[a_{\max} + a_s]^2}{a_{\min}^4} \bigg\} a^2 + a_x^2 \, dx$$

$$\leq \Gamma_0 \int_R \bigg\{ u + \frac{(u_s - c)[(a + a_s)^2 - a_s^2]}{(a + a_s)^2} \bigg\}^2 + a^2 + a_x^2 \, dx,$$

where

(5.20)
$$\Gamma_0 \equiv \max \bigg\{ 2, \sup_R \bigg[1 + 2 \frac{(u_s - c)^2[a_{\max} + a_s]^2}{a_{\min}^4} \bigg] \bigg\} > 0,$$

$$0 < a_{\min} \leq a + a_s(x) \leq a_{\max} < \infty; \ a_s(x) > 0.$$

The inequality (5.20) is the assumption that the tube does not collapse and that it remains finite for all time $t \geq 0$.

To proceed further we must introduce appropriate convexity assumptions on the strain energy function. It follows from the mean value theorem that

(5.21) $\quad \widetilde{W} - W^s - W_a^s a - W_{a_x}^s a_x$

$$= \frac{1}{2} \bigg(a \frac{\partial}{\partial a_s} + a_x \frac{\partial}{\partial a_{s_x}} \bigg)^2 \widehat{W}(a_s + \theta a, a_{s_x} + \theta a_x),$$

for some $\theta \in (0, 1)$ where

$$\widehat{W}(a_s + \theta a, a_{s_x} + \theta a_x) \equiv W(a_s + \theta a, (1 + e)(1 + (a_s + \theta a)_x^2)^{1/2}).$$

Let us suppose that the function $\widehat{W} = \widehat{W}(x, y)$ satisfies the convexity estimates

(5.22) $\qquad 0 < \alpha_1 < \widehat{W}_{xx} < \alpha_2 < \infty,$

(5.23) $\qquad -\infty < \beta_1 < (\widehat{W}_{xy})^2 - \widehat{W}_{xx}\widehat{W}_{yy} < \beta_2 < 0,$

for all (x, y). It follows that the minimum and maximum eigenvalues (over all (x, y)), denoted λ_1 and λ_2, respectively, of the real symmetric matrix

$$(5.24) \qquad \begin{bmatrix} \widehat{W}_{xx} & \widehat{W}_{xy} \\ \widehat{W}_{xy} & \widehat{W}_{yy} \end{bmatrix},$$

are strictly positive and bounded. Thus, from the theory of quadratic forms, we have

$$(5.25) \quad 0 < \frac{\lambda_1}{2}(a^2 + a_x^2) \le \widetilde{W} - W^s - W_a^s a - W_{a_x}^s a_x \le \frac{\lambda_2}{2}(a^2 + a_x^2),$$

for all perturbations.

We will need to introduce additional conditions on these eigenvalues. The second term in (5.18) can be estimated as follows

$$\frac{(u_s - c)^2[(a+a_s)^2 - a_s^2]^2}{2(a+a_s)^2} = \frac{(u_s - c)^2[a + a_s + a_s]^2 a^2}{2(a+a_s)^2}$$
$$\le \frac{(u_s - c)^2[a_{\max} + a_s]^2 a^2}{2a_{\min}^2}.$$

It therefore follows that

$$(5.26) \quad \left\{ -\frac{(u_s - c)^2[a_{\max} + a_s]^2}{2a_{\min}^2} + \frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty + \lambda_1 \right\} a^2$$
$$\le -\frac{(u_s - c)^2[(a+a_s)^2 - a_s^2]^2}{2(a+a_s)^2} + \left(\frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty + \lambda_1 \right) a^2$$

This inequality will be bounded below by zero provided

$$(5.27) \qquad \lambda_1 > \sup_R \left\{ \frac{(u_s - c)^2[a_{\max} + a_s]^2}{2a_{\min}^2} + \frac{c^2}{2} + W_a^\infty - \frac{c^2}{2a_s^4} \right\}.$$

Assuming (5.27) holds, it follows from (5.19) that

(5.28)

$$\|(u,a)\|^2 \le \Gamma_1 \int_R \frac{(a+a_s)^2}{2} \left\{ u + \frac{(u_s - c)[(a+a_s)^2 - a_s^2]}{(a+a_s)^2} \right\}^2$$
$$\left\{ -\frac{(u_s - c)^2[a_{\max} + a_s]^2}{2a_{\min}^2} + \frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty + \lambda_1 \right\} a^2$$
$$+ \lambda_1 a_x^2 \, dx$$
$$\le \Gamma_1 \mathcal{N}(u, a),$$

where

$$\Gamma_1 \equiv \frac{\Gamma_0}{\Gamma_2} > 0,$$

$$\Gamma_2 \equiv \min\{\lambda_1, a_{\min}^2/2, \Gamma_3\} > 0,$$

$$\Gamma_3 \equiv \inf_R \left\{ -\frac{(u_s - c)^2[a_{\max} + a_s]^2}{2a_{\min}^2} + \frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty + \lambda_1 \right\} > 0.$$

The upper estimate is straightforward. We have

$$
\begin{aligned}
(5.29)\qquad \mathcal{N}(u,a) &= \int_R \frac{(a + a_s)^2}{2} \left\{ u + \frac{(u_s - c)[(a + a_s)^2 - a_s^2]}{(a + a_s)^2} \right\}^2 \\
&\quad - \frac{(u_s - c)^2[(a + a_s)^2 - a_s^2]^2}{2(a + a_s)^2} \\
&\quad + \left(\frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty \right) a^2 \\
&\quad + 2[\widetilde{W} - W^s - W_a^s a - W_{a_x}^s a_x]\, dx \\
&\leq \int_R a_{\max}^2 \left\{ u^2 + \frac{(u_s - c)^2[a_{\max} + a_s]^2 a^2}{a_{\min}^4} \right\} \\
&\quad + \left(\frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty + \lambda_2 \right) a^2 + \lambda_2 a_x^2 \, dx \\
&\leq \Gamma_4 \|(u,a)\|^2,
\end{aligned}
$$

where

$$\Gamma_4 = \max\{\lambda_2, \Gamma_5, a_{\max}^2\} > 0,$$

$$\Gamma_5 = \sup_R \left\{ \frac{a_{\max}^2(u_s - c)^2[a_{\max} + a_s]^2}{a_{\min}^4} + \frac{c^2}{2a_s^4} - \frac{c^2}{2} - W_a^\infty + \lambda_2 \right\} > 0.$$

Putting the estimates (5.28) and (5.29) together and exploiting the invariance of $\mathcal{N}(u,a)$ we have

$$\|(u,a)\|^2 \leq \Gamma_1 \mathcal{N}(u,a) = \Gamma_1 \mathcal{N}(u_0, a_0) \leq \Gamma_1 \Gamma_4 \|(u_0, a_0)\|^2,$$

where $(u_0, a_0) = (u,a)_{t=0}$. Consequently, for every $\varepsilon > 0$

$$\|(u_0, a_0)\| < \varepsilon/(\Gamma_1 \Gamma_4)^{1/2} \implies \|(u,a)\| < \varepsilon,$$

for all $t > 0$. We have therefore proved our final result.

Theorem 8. *Suppose the strain energy function W written in the form*

$$W(x, (1 + e)(1 + y^2)^{1/2}) \equiv \widehat{W}(x, y),$$

satisfies (5.22) and (5.23) with λ_1 and λ_2 the positive global (over all $(x, y) \in \mathbf{R}^2$) minimum and maximum eigenvalues of the Hessian matrix (5.24) with λ_1 satisfying (5.27). Then the steadily-travelling solution $a_s(x)$ satisfying (4.10) is nonlinearly stable in the sense of Liapunov with respect to the finite-amplitude perturbation norm

$$\|(u, a)\|^2 \equiv \int_R u^2 + a^2 + a_x^2 \, dx,$$

provided $0 < a_{\min} \leq a(x, t) + a_s(x) \leq a_{\max} < \infty$ for all $t \geq 0$.

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