

FINITE-AMPLITUDE PERTURBATIONS AND MODULATIONAL INSTABILITY OF A STABLE GEOSTROPHIC FRONT

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The finite-amplitude evolution of neutral perturbations to the Cushman-Roisin frontal geostrophic model for a simple upwelling front with spatially varying potential vorticity is determined. It is shown that the sinuous and varicose modes are governed by the “bright” and “dark” NLS equations, respectively. This implies that the sinuous modes can exhibit Benjamin-Feir instability (while the varicose modes do not), suggesting the possibility that envelope solitons can form on a frontal outcropping. Exploiting the underlying Hamiltonian structure, it is nevertheless shown that all monotonic parallel front solutions of the Cushman-Roisin model are nonlinearly stable in the sense of Liapunov.

Keywords: Frontal dynamics; nonlinear waves; modulational instability; Hamiltonian dynamics; physical oceanography

1. INTRODUCTION

In principle, quasigeostrophic theory is not applicable in determining the evolution of oceanographic fronts as this theory implicitly assumes that the scaling associated with the dynamic deflections of the isopycnals is small in comparison with the ambient scale depth. For fronts, the depth variation of the front is often on the order of the local

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scale depth itself. In fact, isopycnals associated with fronts may intersect either the surface of the ocean or the ocean bottom.

Nevertheless, quasigeostrophic theory has been able to explain, at least qualitatively, some aspects of frontal dynamics. For example, Orlandi (1968) and Smith (1976) were able to reproduce some baroclinic instability characteristics of fronts using a quasigeostrophic model. The limitations of quasigeostrophic theory became apparent, however, in experiments done by Griffiths and Linden (1981) in which there were significant differences between the predictions of quasigeostrophic theory and the observations.

Griffiths *et al.* (1982) developed a frontal model that was not quasigeostrophic. Their reduced-gravity model predicted instability even when there was no extremum in the potential vorticity as is required in quasigeostrophic theory (see, e.g., LeBlond and Mysak, 1978). Their model, however, still could not explain the instabilities observed in their experiments. Subsequently, Killworth and Stern (1982) included the effects of a second layer and showed that this layer had significant effects on the instability characteristics.

In a series of articles, Paldor (1983a,b, 1986) considered perturbations to a zero potential vorticity front described by the reduced-gravity equations. Paldor showed that zero potential vorticity solutions of the shallow water equations were neutrally stable and he conjectured that it was small deviations from constant potential vorticity which drove the instabilities seen in laboratory experiments (e.g., Griffiths *et al.*, 1982) as well as in oceanographic observations. Here, we attempt to address this issue by determining the finite amplitude evolution of perturbations to a geostrophic front with a simple spatially varying potential vorticity.

Cushman-Roisin (1986) derived a reduced-gravity model which included the leading order ageostrophic terms associated with the nonlinear advective terms in the momentum equations. Using this model Cushman-Roisin solved the linear stability problem for a simple monotonic front with the property that the potential vorticity varied spatially and showed the front to be linearly stable. This result suggested that baroclinic processes and/or higher order ageostrophic effects may be crucial in the transition to instability. Similar remarks have been made by, for example, Benilov (1992, 1994) and Benilov and Cushman-Roisin (1994).

Swaters (1993) extended the Cushman-Roisin model to include baroclinic processes and a background vorticity gradient associated with a sloping bottom. He considered the linear stability problem for the reduced gravity limit of his model and showed that all fronts are linearly stable in the sense of Liapunov by exploiting the underlying Hamiltonian structure of the model.

Experiments, however, have shown that finite amplitude effects are important in the stability of frontal flows (e.g., Griffiths and Linden, 1981). Swaters (1993) examined the nonlinear stability problem exploiting the Hamiltonian formulation of the model. He was able to establish, using the pseudo-energy, conditions for nonlinear stability but only under certain mathematical restrictions (a Poincaré inequality was required). Karsten and Swaters (1996) re-examined the stability problem and were able to establish conditions for nonlinear stability without requiring a Poincaré inequality using the pseudo-momentum.

The principal purpose of this paper is to develop a finite amplitude normal mode theory for the perturbations of the simple monotonic front profile examined by Cushman-Roisin (1986) within the context of the reduced gravity limit of the Swaters (1993) model. It will be shown that the upwelling front examined by Cushman-Roisin (1986) is nonlinearly stable in the sense of Liapunov.

Even though the frontal distortions examined in this paper correspond to neutral perturbations of the front, we believe that it is important to understand the full range of dynamical responses (including stable ones) a perturbed frontal flow can exhibit. Here, we show, perhaps surprisingly, that the finite-amplitude evolution of neutral sinuous (even in the cross-front direction) and varicose (odd in the cross-front direction) perturbations are qualitatively different. The sinuous and varicose amplitudes are governed by the *bright* and *dark* nonlinear Schrödinger (NLS) equations, respectively.

This fact has an important implication on the secondary instability properties of a perturbed frontal outcropping. It is well known (see, e.g., Benjamin and Feir, 1967, Newell, 1985; Craik, 1985) that a monochromatic wave is modulationally unstable (stable) if the wave amplitude is governed by the bright (dark) NLS equation. Thus our results suggest, at least for the front considered here, that the sinuous branch of neutral perturbations exhibit Benjamin-Feir or modulational instability.

It is important to emphasize that the underlying nonlinear stability of the front examined here and the modulation instability of the sinuous perturbation amplitudes do not contradict each other. A modulationally unstable monochromatic wave eventually evolves into a sequence of amplitude bounded NLS solitons and dispersive wave tail (Newell, 1985). From the point of view of a mathematical proof, the amplitude of the solitons and wave tail can be made as small as one wants by controlling the initial perturbation norm.

An issue which initially might seem to severely restrict our results is that our calculations are specific to a very simple front with a single outcropping in which the along front geostrophic velocity is unshered. However, as we show, all fronts with a monotonic thickness profile are nonlinearly stable in the sense of Liapunov according to the Cushman-Roisin model. Thus we believe that the results presented here describe, at least qualitatively, what one can expect for the finite amplitude dynamics of neutral wave packets for the entire class of stable monotonic fronts with nonuniform potential vorticity.

The paper is set out as follows. In Section 2 the model is derived and in Section 3 the Hamiltonian formulation is briefly introduced and used to establish stability. In Section 4 we determine the finite-amplitude evolution of neutral perturbations for a linearly varying upwelling front. In Section 5 we discuss the stability of the Stokes wave solution for the perturbation amplitude. In Section 6 we present our main conclusions.

2. MODEL FORMULATION

The nondimensional equations for the upper layer, assuming a f_0 -plane, reduced-gravity shallow water system (see Fig. 1), can be written in the form

$$\left. \begin{aligned} \delta^2 \mathbf{u}_t + \delta \mathbf{u} \cdot \nabla \mathbf{u} + \hat{\mathbf{e}}_3 \times \mathbf{u} + \nabla h &= \mathbf{0}, \\ \delta h_t + \nabla \cdot (\mathbf{u}h) &= 0, \end{aligned} \right\} \quad (2.1)$$

where \mathbf{u} is the horizontal velocity vector, h is the thickness of the front, (x,y) are the spatial coordinates and t is time. Alphabetical subscripts, except where indicated, represent partial differentiation.

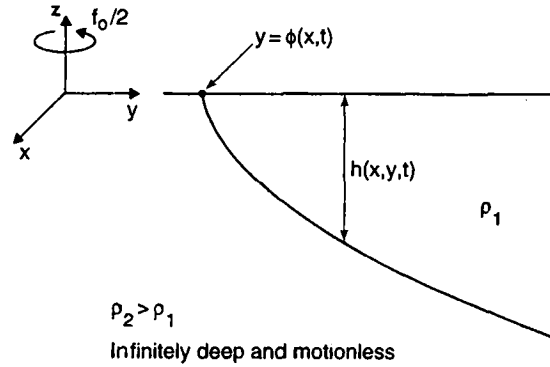


FIGURE 1 Geometry of the reduced gravity model used in this paper.

The dimensional (asterisked) variables are related to the nondimensional variables by the scalings

$$\left. \begin{aligned} (x^*, y^*) &= L(x, y), & t^* &= t/f_0\delta^2, \\ (u^*, v^*) &= (f_0L\delta)(u, v), & h^* &= h_0h, \end{aligned} \right\} \quad (2.2)$$

where $L = (gh_0/\delta)^{1/2}/f_0$, h_0 is a representative scale height for the front, and δ is a small parameter. These scalings may be thought of as corresponding to a subinertial approximation in which the horizontal length scale is larger than the internal deformation radius while allowing for finite-amplitude interface deflections.

The position of the outcropping on the surface is given by $y = \phi(x, t)$. Since the thickness of the front must be zero on the outcropping, the nondimensional dynamic boundary condition is simply

$$h(x, y, t) = 0 \quad \text{on} \quad y = \phi(x, t). \quad (2.3)$$

The nondimensional kinematic boundary condition is

$$v = \delta\phi_t + u\phi_x \quad \text{on} \quad y = \phi(x, t). \quad (2.4)$$

We also require that the velocity field be bounded as y gets large, that is

$$|\mathbf{u}| < \infty \quad \text{as} \quad y \rightarrow \infty, \quad (2.5)$$

if the domain is unbounded in the positive y -direction and we assume, without loss of generality, that the front is confined to $y > \phi(x, t)$.

The Cushman-Roisin model corresponds to assuming $0 < \delta \ll 1$ and constructing a solution in the form

$$(\mathbf{u}, h) \simeq (\mathbf{u}^{(0)}, h^{(0)}) + \delta(\mathbf{u}^{(1)}, h^{(1)}) + \dots \quad (2.6)$$

and

$$\phi \simeq \phi^{(0)} + \delta\phi^{(1)} + \dots \quad (2.7)$$

yielding

$$h_t + J\left(h\Delta h + \frac{1}{2}\nabla h \cdot \nabla h, h\right) = 0, \quad (2.8)$$

where $J(A, B) = A_x B_y - A_y B_x$, with the boundary conditions

$$\left. \begin{aligned} h(x, y, t) &= 0 \quad \text{on } y = \phi(x, t), \\ |\nabla h| &< \infty \quad \text{as } y \rightarrow \infty, \end{aligned} \right\} \quad (2.9)$$

where we have deleted, for notational convenience, the (0) superscript.

3. HAMILTONIAN FORMULATION AND NONLINEAR STABILITY

Here we exploit the underlying Hamiltonian structure of the model and show that all steady monotonic fronts are nonlinearly stable in the sense of Liapunov. A system of partial differential equations is Hamiltonian if it can be written in the form (Olver, 1982)

$$\mathbf{q}_t = D \frac{\delta H}{\delta \mathbf{q}}, \quad (3.1)$$

where \mathbf{q} is a column vector of n independent variables, $H(\mathbf{q})$ is a conserved (Hamiltonian) functional, $\delta H/\delta \mathbf{q}$ are the Euler derivatives of H with respect to \mathbf{q} , and D is a matrix of differential operators. The Poisson bracket of a Hamiltonian system, defined by (Morrison, 1982)

$$[F, G] \equiv \left\langle \frac{\delta F}{\delta \mathbf{q}}, D \frac{\delta G}{\delta \mathbf{q}} \right\rangle, \quad (3.2)$$

where F and G are arbitrary functionals of \mathbf{q} , must satisfy the properties of skew-symmetry, associative and distributive laws and the Jacobi identity.

It is straightforward to verify that the frontal model (2.8) and (2.9) can be written as the scalar Hamiltonian system

$$q = h, \quad (3.3)$$

$$H(q) = -\frac{1}{2} \int_{\Omega} \int_{\Omega} h \nabla h \cdot \nabla h \, dx dy, \quad (3.4)$$

$$D(*) = J(q, *) \quad (3.5)$$

with the Poisson bracket

$$[F, G] = \int_{\Omega} \int_{\Omega} \frac{\delta F}{\delta h} J \left(h, \frac{\delta G}{\delta h} \right) dx dy \quad (3.6)$$

and where we take the spatial domain Ω to be the periodic channel

$$\Omega = \{(x, y) \mid -x_0 < x < x_0, \quad \phi(x, t) < y < (L) \leq \infty\}. \quad (3.7)$$

Verification of the required algebraic properties for the Poisson bracket can be found in Slomp (1995).

While it is possible to construct a Hamiltonian-based stability theory for arbitrary steady solutions to the model, it is straightforward to show, based on a direct application of Andrews' theorem (Andrews, 1984), that the only class of flows in the channel domain (3.7) which can be Arnold-stable are parallel shear flows. In our context these are frontal solutions for which the undisturbed outcroppings, if any, are parallel to the x -axis and $h = h_0(y)$.

Swaters (1993) has shown that all steady solutions to (2.8) are linearly stable in the sense of Liapunov. Here we show, by using a variational principle based on the x -direction impulse invariant, that all monotonic parallel frontal flows are nonlinearly stable in the sense of Liapunov.

The linear stability problem is formed by assuming

$$h(x, y, t) = h_0(y) + \delta h(x, y, t), \quad (3.8)$$

where $h_0(y)$ is the steady parallel shear flow. Inserting (3.8) into (2.8) gives the linear stability equation

$$\delta h_t + (h_{0y})^2 \delta h_{yx} + h_0 h_{0y} \Delta \delta h_x - h_{0y} h_{0yy} \delta h_x - h_0 h_{0yy} \delta h_x = 0. \quad (3.9)$$

Steady solutions of the form $h_0 = h_0(y)$ satisfy the first-order necessary conditions for an extremum of the constrained linear momentum functional (Karsten and Swaters, 1996)

$$\mathcal{M}(h) = M(h) + C(h), \quad (3.10)$$

where $M(h)$ is the x -direction impulse functional

$$M(h) = - \int \int_{\Omega} y h \, dx dy, \quad (3.11)$$

and $C(h)$ is the Casimir (see, e.g., Shepherd, 1990) given by

$$C(h) = \int \int_{\Omega} [\Phi(h) - \Phi(0)] \, dx dy, \quad (3.12)$$

where $\Phi(h)$ is defined so that

$$\Phi'(h_0(y)) = y, \quad (3.13)$$

that is $\Phi'(\cdot)$ is the inverse function associated with $h_0(y)$.

It is straightforward to show that the second variation of \mathcal{M} evaluated at the steady solution, given by

$$\delta^2 \mathcal{M}(h_0) = \int \int_{\Omega_0} \Phi''(h_0) (\delta h)^2 \, dx dy, \quad (3.14)$$

is an invariant of (3.9). Thus, under the conditions

$$0 < \inf_{\Omega_0} \Phi''(h_0) = \inf_{\Omega_0} \frac{1}{h_0'(y)} < \infty \quad (3.15)$$

or

$$-\infty < \sup_{\Omega_0} \Phi''(h_0) = \sup_{\Omega_0} \frac{1}{h'_0(y)} < 0, \quad (3.16)$$

$\delta^2 \mathcal{M}(h_0)$ is definite and linear stability in the sense of Liapunov with respect to the energy norm

$$\|\delta h\|^2 = \int \int_{\Omega} h^2 dx dy \quad (3.17)$$

can be proven.

Using the nonlinear invariant

$$\mathcal{L}(h) = M(h + h_0) - M(h_0) + C(h + h_0) - C(h_0), \quad (3.18)$$

where h is a finite amplitude perturbation, it can be shown (see, e.g., Karsten and Swaters 1996) that $h_0(y)$ is nonlinearly stable in the sense of Liapunov with respect to the energy norm if

$$-\infty < \alpha_1 < \frac{1}{h'_0(\xi)} < \beta_1 < 0 \quad (3.19)$$

or

$$0 < \alpha_2 < \frac{1}{h'_0(\xi)} < \beta_2 < \infty \quad (3.20)$$

for all ξ and where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are finite real constants. Therefore, all parallel fronts with monotonic thickness profiles are nonlinearly stable in the sense of Liapunov.

4. FINITE-AMPLITUDE EVOLUTION

Cushman-Roisin (1986) showed that if $h_0(y) = \alpha y$ where $\alpha > 0$ and $y > 0$, then the linear stability problem given by

$$h_t + \alpha^2 y h_{xxx} + \alpha^2 y h_{yyx} + \alpha^2 h_{yx} = 0, \quad (4.1)$$

where h is an infinitesimal perturbation, has the neutral mode solution given by

$$h(x, y, t) = AL_n(2ky)\exp(-ky)\exp[ik(x - ct)] + c.c., \quad (4.2)$$

where A is a free amplitude constant, L_n is the Laguerre polynomial of order n , k is the positive alongshore wavenumber, $c.c.$ is the complex conjugate and c is the along-front phase velocity satisfying the dispersion relationship

$$c = -k\alpha^2(2n + 1) \quad (4.3)$$

with n any non-negative integer.

4.1. The Nonlinear Problem

The nonlinear stability problem for the simple wedge front $h_0 = \alpha y$ can be written in the form

$$\begin{aligned} & h_t + \alpha^2 h_{xxx} + \alpha y h_y h_{xxx} + \alpha h h_{xxx} + \alpha^2 y h_{xxx} \\ & + \alpha y h_y h_{yyx} + \alpha h h_{yyx} + h h_y h_{yyx} + \alpha h_x h_{xx} + h_x h_{xx} h_y \\ & + \alpha^2 h_{yx} + 2\alpha h_y h_{yx} + (h_y)^2 h_{yx} - \alpha y h_{xxy} h_x - h h_{xxy} h_x \\ & - \alpha h_{yy} h_x - h_y h_{yy} h_x - h h_{yyy} h_x - h_x h_{xy} h_x = 0, \end{aligned} \quad (4.4)$$

where h is a finite amplitude perturbation.

If we assume that the undisturbed outcropping occurs at $y=0$, then the perturbation thickness must satisfy

$$\alpha y + h = 0, \quad \text{on } y = \phi(x, t), \quad (4.5)$$

where ϕ is the perturbed location of the outcropping. This expression can be Taylor expanded to yield

$$\alpha\phi(x, t) + h + h_y\phi(x, t) + \text{h.o.t.} = 0 \quad \text{on } y = 0. \quad (4.6)$$

If L is finite or infinite we require, respectively,

$$\left. \begin{aligned} h(x, L, t) &= 0 \quad \text{on } y = L, \\ |\nabla h|(x, L, t) &< \infty. \end{aligned} \right\} \quad (4.7)$$

However, henceforth we will assume that L is infinite.

4.2. Multiple-Scale Analysis

In order to account for the space and time scales for which the nonlinear terms make an $O(1)$ contribution to the evolution of the perturbation field we introduce the rescaling and the slow space and time variables

$$h \rightarrow \varepsilon h, \quad \phi \rightarrow \varepsilon \phi, \quad (4.8)$$

$$X = \varepsilon x, \quad (T, \tau) = (\varepsilon t, \varepsilon^2 t), \quad (4.9)$$

where ε is a small perturbation amplitude parameter.

Introduction of (4.8) and (4.9) into (4.4) leads to

$$\begin{aligned} & h_t + \varepsilon h_T + \varepsilon^2 h_\tau + \alpha^2 y (h_{xxx} + \varepsilon h_{xxX} + 3\varepsilon^2 h_x X X) \\ & + \varepsilon \alpha y h_y (h_{xxx} + 3\varepsilon h_{xxX}) + \varepsilon \alpha h (h_{xxx} + 3\varepsilon h_{xxX}) + \varepsilon^2 h h_y h_{xxx} \\ & + \alpha^2 y (h_{yyx} + \varepsilon h_{yyX}) + \varepsilon \alpha y h_y (h_{yyx} + \varepsilon h_{yyX}) \\ & + \varepsilon \alpha h (h_{yyx} + \varepsilon h_{yyX}) + \varepsilon^2 h h_y h_{yyx} \\ & + \varepsilon \alpha (h_x + \varepsilon h_X) (h_{xx} + 2\varepsilon h_x X) + \varepsilon^2 h_y h_x h_{xx} \\ & + \alpha^2 (h_{yx} + \varepsilon h_{yX}) + 2\varepsilon \alpha h_y (h_{yx} + \varepsilon h_{yX}) + \varepsilon^2 (h_y)^2 h_{yx} \\ & - \varepsilon \alpha y (h_{xxy} + 2\varepsilon h_x X y) (h_x + \varepsilon h_X) - \varepsilon^2 h h_{xxy} h_x \\ & - \varepsilon \alpha h_{yy} (h_x + \varepsilon h_X) - \varepsilon^2 h_y h_{yy} h_x - \varepsilon \alpha y h_{yyy} (h_x + \varepsilon h_X) \\ & - \varepsilon^2 h h_{yyy} h_x - \varepsilon^2 (h_x)^2 h_{xy} + O(\varepsilon^3) = 0 \end{aligned} \quad (4.10)$$

with the outcropping condition

$$\alpha \phi + h + \varepsilon h_y \phi + O(\varepsilon^2) = 0 \quad \text{on } y = 0. \quad (4.11)$$

Substitution of the straightforward asymptotic expansion

$$(h, \phi) \simeq (h, \phi)^{(0)} + (h, \phi)^{(1)} + (h, \phi)^{(2)} + \dots \quad (4.12)$$

into (4.10) and (4.11) yields the $O(1)$ problem

$$h_t^{(0)} + \alpha^2 y h_{xxx}^{(0)} + \alpha^2 y h_{yyx}^{(0)} + \alpha^2 h_{yx}^{(0)} = 0, \quad (4.13)$$

$$\alpha \phi^{(0)} + h^{(0)} = 0 \quad \text{on } y = 0. \quad (4.14)$$

The bounded normal mode solution (Cushman-Roisin, 1986) is given by

$$\begin{aligned} h^{(0)} &= A \exp(-ky) L_n(2ky) \exp[ik(x-ct)] + c.c. \\ &\equiv A \psi(y) \exp[ik(x-ct)] + c.c., \end{aligned} \quad (4.15)$$

where $A = A(X, T, \tau)$ is the slowly varying amplitude and where c is given by the dispersion relation (4.3). The $O(1)$ outcropping location is given by

$$\phi^{(0)} = -\frac{A}{\alpha} \exp[ik(x-ct)] + c.c.. \quad (4.16)$$

We call the even and odd n neutral solutions sinuous and varicose modes, respectively.

The $O(\varepsilon)$ problem is given by

$$\begin{aligned} h_i^{(1)} + \alpha^2 y h_{xxx}^{(1)} + \alpha^2 y h_{yyx}^{(1)} + \alpha^2 h_{yx}^{(1)} = \\ - h_T^{(0)} - 3\alpha^2 y h_{xxX}^{(0)} - \alpha y h_y^{(0)} h_{xxx}^{(0)} - \alpha h^{(0)} h_{xxx}^{(0)} - \alpha^2 y h_{yyX}^{(0)} \\ - \alpha y h_y^{(0)} h_{yyx}^{(0)} - \alpha h^{(0)} h_{yyx}^{(0)} - \alpha h_x^{(0)} h_{xx}^{(0)} - \alpha^2 h_{yX}^{(0)} - 2\alpha h_y^{(0)} h_{yx}^{(0)} \\ + \alpha y h_{xy}^{(0)} h_x^{(0)} + \alpha h_{yy}^{(0)} h_x^{(0)} + \alpha y h_{yyy}^{(0)} h_x^{(0)} \end{aligned} \quad (4.17)$$

with the $O(\varepsilon)$ outcropping location determined by

$$\phi^{(1)} = \frac{1}{\alpha^2} h_y^{(0)} h^{(0)} - \frac{1}{\alpha} h^{(1)} \quad \text{on } y = 0. \quad (4.18)$$

The solution (see Appendix A for details) to the $O(\varepsilon)$ problem may be written in the form

$$\begin{aligned} h^{(1)}(x, y, t) &= \frac{iA\xi}{k} \exp(-ky) [ky L_n(2ky) + n L_{n-1}(2ky)] \exp(i\theta) \\ &\quad + \frac{A^2 k}{\alpha} \exp(-2ky) \Psi_n(2ky) \exp(2i\theta) + c.c. \\ &\quad + b(y; \xi, \tau), \end{aligned} \quad (4.19)$$

where $\theta = k(x-ct)$, $\xi = X + \alpha^2 k(2n+1)T$, $\Psi_n(2ky)$ is given by (A.18) and where $b(y; \xi, \tau)$ is a mean flow generated by the interaction of the fundamental mode with itself and is determined in the $O(\varepsilon^2)$ problem.

It is shown in Appendix A that, as expected, solvability conditions associated with (4.17) force $A = A(\xi, \tau)$, that is, to this order, A is invariant following the group velocity $c_g \equiv -2k\alpha^2(2n+1)$.

The corresponding $O(\varepsilon)$ perturbation outcropping is given by

$$\begin{aligned} \phi^{(1)} = & \frac{-k[A\exp(i\theta) + A^*\exp(-i\theta)]^2(1+2n)}{\alpha^2} \\ & - \frac{in[A_\xi\exp(i\theta) - A_\xi^*\exp(-i\theta)]}{k\alpha} \\ & \frac{k[A^2\exp(2i\theta) + A^{*2}\exp(-2i\theta)]}{\alpha^2} \Psi(0) - \frac{b(0; \xi, \tau)}{\alpha}. \end{aligned} \quad (4.20)$$

The $O(\varepsilon^2)$ problem is given by

$$\begin{aligned} h_i^{(2)} + \alpha^2 y h^{(2)} h_{xxx} + \alpha^2 y h_{yx}^{(2)} + \alpha^2 h_{yx}^{(2)} = & -h_T^{(1)} - h_\tau^{(0)} - 3\alpha^2 y h_{xxX}^{(1)} \\ & - 3\alpha^2 y h_{xXX}^{(0)} - \alpha y h_y^{(0)} h_{xxx}^{(1)} - \alpha y h_y^{(1)} h_{xxx}^{(0)} - 3\alpha y h_y^{(0)} h_{xxX}^{(0)} - \alpha h^{(0)} h_{xxx}^{(1)} \\ & - \alpha h^{(1)} h_{xxx}^{(0)} - 3\alpha h^{(0)} h_{xxX}^{(0)} - h^{(0)} h_y^{(0)} h_{xxx}^{(0)} - \alpha^2 y h_{yyX}^{(1)} - \alpha y h_y^{(0)} h_{yyx}^{(1)} \\ & - \alpha y h_y^{(1)} h_{yyx}^{(0)} - \alpha y h_y^{(0)} h_{yyX}^{(0)} - \alpha h^{(0)} h_{yyx}^{(1)} - \alpha h^{(1)} h_{yyx}^{(0)} - \alpha h^{(0)} h_{yyX}^{(0)} \\ & - h^{(0)} h_y^{(0)} h_{yyx}^{(0)} - \alpha h_x^{(1)} h_{xx}^{(0)} - \alpha h_x^{(0)} h_{xx}^{(1)} - 2\alpha h_x^{(0)} h_{xX}^{(0)} - \alpha h_X^{(0)} h_{xx}^{(0)} \\ & - h_x^{(0)} h_{xx}^{(0)} h_y^{(0)} - \alpha^2 h_{yX}^{(1)} - 2\alpha h_y^{(1)} h_{yx}^{(0)} - 2\alpha h_y^{(0)} h_{yX}^{(1)} - 2\alpha h_y^{(0)} h_{yX}^{(0)} \\ & - \left(h_y^{(0)}\right)^2 h_{yx}^{(0)} + \alpha y h_{xxy}^{(1)} h_x^{(0)} + \alpha y h_{xxy}^{(0)} h_x^{(1)} + 2\alpha y h_{xXy}^{(0)} h_x^{(0)} + \alpha y h_{xxy}^{(0)} h_X^{(0)} \\ & + h^{(0)} h_{xxy}^{(0)} h_x^{(0)} + \alpha h_{yy}^{(1)} h_x^{(0)} + \alpha h_{yy}^{(0)} h_x^{(1)} + \alpha h_{yy}^{(0)} h_X^{(0)} + h_y^{(0)} h_{yy}^{(0)} h_x^{(0)} \\ & + \alpha y h_{yyy}^{(1)} h_x^{(0)} + \alpha y h_{yyy}^{(0)} h_x^{(1)} + \alpha y h_{yyy}^{(0)} h_X^{(0)} + h^{(0)} h_{yyy}^{(0)} h_x^{(0)} + \left(h_x^{(0)}\right)^2 h_{xy}^{(0)}. \end{aligned} \quad (4.21)$$

It is not necessary to solve for $h^{(2)}$. Instead solvability conditions associated with (4.21) establish equations which determine the evolution of the envelope amplitude $A(\xi, \tau)$ and the mean flow $b(y; \xi, \tau)$. These solvability conditions (see Appendix B) determine that the free amplitude, $A(\xi, \tau)$, must satisfy the nonlinear Schrödinger (NLS) equation

$$A_\tau = \frac{i\omega''}{2} A_{\xi\xi} + ik^4 \beta_n |A|^2 A, \quad (4.22)$$

where

$$\omega'' = -2\alpha^2(2n + 1) < 0, \quad (4.23)$$

and where β_n , the nonlinear coefficient, depends on n . This coefficient is tabulated in Table II for the first few n .

5. STOKES WAVE SOLUTION AND MODULATIONAL INSTABILITY

We see in Table II that the even and odd, with respect to the index n , β_n coefficients are negative or positive definite, respectively. Although we were unable to verify this property for all n (the calculations become increasing costly), this property held true for all the n which we checked. In the jargon of soliton theory (see, e.g., Newell, 1985 or Drazin and Johnson, 1989), if the product $\omega''k^4\beta_n$ is positive or

TABLE I Table of the series coefficients a_k

n	0	1	2	3	4	5
a_0	0	-4	$\frac{28}{4}$	$-\frac{856}{77}$	$\frac{2024}{221}$	$-\frac{80772}{4807}$
a_1	-	-2	$-\frac{192}{5}$	$\frac{292}{77}$	$-\frac{15924}{221}$	$\frac{75054}{4807}$
a_2	-	-	$\frac{16}{5}$	$\frac{1388}{77}$	$\frac{15364}{221}$	$\frac{343660}{4807}$
a_3	-	-	$-\frac{1}{3}$	$-\frac{359}{77}$	$-\frac{4917}{221}$	$-\frac{1010665}{14421}$
a_4	-	-	-	$\frac{7}{11}$	$\frac{3821}{663}$	$\frac{124494}{48076}$
a_5	-	-	-	$-\frac{1}{30}$	$-\frac{1135}{1328}$	$-\frac{20063}{4807}$
a_6	-	-	-	-	$\frac{10}{153}$	$\frac{3671}{3933}$
a_7	-	-	-	-	$-\frac{1}{304}$	$-\frac{865}{10488}$
a_8	-	-	-	-	-	$\frac{13}{3312}$
a_9	-	-	-	-	-	$-\frac{1}{12960}$

TABLE II Table of values for the nonlinear coefficient β_n

n	β_n
0	-0.64
1	28.07
2	-27.03
3	138.41
4	-137.21

negative definite, then (4.22) is called the “bright” or “dark” NLS equation, respectively.

Bright NLS solitons correspond to solutions of the NLS equation which decay to zero at infinity, that is, the maximum amplitude of the soliton is located at the position where the soliton phase variable is zero. These solutions have the spatial structure of a soliton one usually thinks of. The term “bright” is used to indicate that at the center of the soliton the amplitude of the underlying fast phase oscillations is at its maximum.

Dark NLS solitons correspond to solutions of the NLS equation which have minimum amplitude at the position where the soliton phase variable is zero and which increase to a nonzero constant at infinity. These solutions, therefore, have the property that at the center of the soliton the amplitude of the underlying fast phase oscillations is at its minimum, i.e., the magnitude of the fast phase oscillations is suppressed at the center of the soliton hence the term “dark”.

Since ω'' is strictly negative, it follows that the even n or sinuous modes are unstable while the odd n or varicose modes are neutrally stable to side-band perturbations. This is easily seen by examining the linear instability problem associated with the Stokes wave solution associated with (4.22), given by

$$A = A_0 \exp(ik^4 \beta_n A_0^2 \tau), \quad (5.1)$$

where A_0 is a real-valued constant. The Stokes wave solution describes the leading order amplitude correction to the frequency of the underlying normal mode.

If we consider a perturbed Stokes wave solution to (4.22) of the form

$$A = [A_0 + b(\xi, \tau)] \exp(ik^4 \beta_n A_0^2 \tau), \quad (5.2)$$

where $b(\xi, \tau) \exp(ik^4 \beta_n A_0^2 \tau)$ is the complex-valued perturbation, it follows that the linear instability equation can be written in the form

$$b_\tau = \frac{i\omega''}{2} b_{\xi\xi} + ik^4 \beta_n A_0^2 (b + b^*), \quad (5.3)$$

where b^* is the complex conjugate of b .

Introducing the decomposition

$$b = b_R(\xi, \tau) + ib_I(\xi, \tau), \quad (5.4)$$

where b_R and b_I are real-valued, leads to the pair of equations

$$\begin{aligned} \frac{\partial b_R}{\partial \tau} &= -\frac{\omega''}{2} \frac{\partial^2 b_I}{\partial \xi^2}, \\ \frac{\partial b_I}{\partial \tau} &= \frac{\omega''}{2} \frac{\partial^2 b_R}{\partial \xi^2} + 2k^4 \beta_n A_0^2 b_R, \end{aligned}$$

which can be combined together to give

$$\left[\partial_{\tau\tau} + \omega'' k^4 \beta_n A_0^2 \partial_{\xi\xi} + \left(\frac{\omega''}{2} \right)^2 \partial_{\xi\xi\xi\xi} \right] b_I = 0. \quad (5.5)$$

Assuming a normal mode solution to (5.5) of the form

$$b_I = \tilde{b}_I \exp[i(\Lambda\xi - \Omega\tau)] + c.c., \quad (5.6)$$

where Λ and Ω are the perturbation wavenumber and frequency, respectively, leads to the dispersion relation

$$\Omega^2 = -\omega'' \Lambda^2 \left(\beta_n k^4 A_0^2 - \frac{\omega''}{4} \Lambda^2 \right). \quad (5.7)$$

Instability occurs if the frequency satisfies $\Omega^2 < 0$ and since $\omega'' < 0$ this can only occur if $\beta_n < 0$ (we assume a real-valued wavenumber Λ) which is precisely the situation for the sinuous or even n modes. The odd n or varicose modes have $\beta_n > 0$ which implies stability, i.e., $\Omega^2 > 0$. However, the instability is band limited since even if $\beta_n < 0$, only those perturbation wavenumbers satisfying

$$0 < \Lambda^2 < \frac{4\beta_n k^4 A_0^2}{\omega''} \quad (5.8)$$

are unstable. The instability corresponds to nearby or side-band perturbations in the wavenumber spectrum of the underlying normal mode in the original instability problem which are able to extract energy from the dominate monochromatic wave field.

It may seem at first contradictory that we have established the nonlinear stability in the sense of Liapunov for the wedge front $h_0 = \alpha y$ only to determine that the sinuous modes associated with the linear instability problem are modulationally unstable. But this is not a contradiction. As discussed by Newell (1985), the unstable side-bands and the dominate wavenumber develop over time to form envelope solitons which are bounded in time and space. Thus, in the long run, all that is occurring is the monochromatic neutral perturbations re-organize themselves into travelling wave-packet solitons. That is, we have identified the possibility that dispersive disturbances on a stable frontal outcropping can evolve into coherent structures via Benjamin-Feir instability. This does not occur for the varicose modes.

6. SUMMARY

An analytic asymptotic theory examining the finite-amplitude evolution of a wedge-shaped front has been derived by using a filtered reduced-gravity model. This model possesses a noncanonical Hamiltonian structure. Using this Hamiltonian structure all monotonic parallel shear flow steady solutions to this model are found to be nonlinearly stable in the sense of Liapunov with respect to the energy norm.

The nonlinear evolution of a perturbed wedge-shaped front is examined using a multiple scales analysis. It is shown that the amplitude of the neutral perturbation waves was governed by the nonlinear Schrödinger equation. For the varicose modes we showed that the perturbation amplitude is governed by the “bright” NLS equations while for the sinuous modes the perturbation amplitude is governed by the “dark” NLS equation. This implies that the sinuous modes are unstable to nearby side-band perturbations in the wavenumber spectrum while the varicose modes are not. The modulationally unstable sinuous modes will, over time, develop into along-front propagating envelope solitons with maximum amplitude located on outcroppings.

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APPENDIX A

The only required homogeneous solution to the $O(\varepsilon)$ problem, denoted as $h_h^{(1)}$, is the mean flow term

$$h_h^{(1)}(x, y, t) = b(y; X, T, \tau). \quad (\text{A.1})$$

The particular solution to the $O(\varepsilon)$ problem, denoted as $h_p^{(1)}$, is of the form

$$h_p^{(1)} = C(X, T, \tau)G(y)\exp(i\theta) + A^2(X, T, \tau)H(y)\exp(2i\theta) + c.c.. \quad (\text{A.2})$$

Substitution into the $O(\varepsilon)$ problem results in the following two ordinary differential equations

$$\begin{aligned} yG'' + G' - [k^2y - k(2n + 1)]G \\ = (ik\alpha^2C)^{-1}\psi\{-A_T + \alpha^2A_X(2k^2y + k(2n + 1))\}, \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} yH'' + H' - [4k^2y - k(2n + 1)]H \\ = \frac{1}{2\alpha} [3k^2\psi^2 - (\psi')^2 - 2\psi\psi''], \end{aligned} \quad (\text{A.4})$$

where $\psi = \exp(-ky)L_n(2ky)$ and primes denote differentiation with respect to y and where

$$y\psi'' + \psi' - [k^2y - k(2n + 1)]\psi = 0, \quad (\text{A.5})$$

has been used.

Application of the Fredholm Alternative Theorem (see, e.g., Zwillinger, 1989), implies that a solution to (A.3) exists only if

$$\int_0^{\infty} \Phi(y)G_h(y)dy = 0, \quad (\text{A.6})$$

where G_h is a homogeneous solution of (A.3) and $\Phi(y)$ is the right-hand side of (A.3).

Since the homogenous problem associated with (A.3) is identical to (A.5) it follows that

$$G_h(y) = \exp(-ky)L_n(2ky) \equiv \psi(y). \quad (\text{A.7})$$

Thus

$$0 = \int_0^{\infty} \Phi(y)\psi(y)dy \quad (\text{A.8})$$

implies that

$$0 = \int_0^{\infty} \exp(-z)[L_n(z)]^2[-A_T + k\alpha^2zA_X + k\alpha^2(2n+1)A_X]dz, \quad (\text{A.9})$$

where the change of variable $z = 2ky$ has been introduced. This integral can be evaluated (for details see Slomp, 1995) to yield

$$A_T = 2\alpha^2k(1-2n)A_X = 0, \quad (\text{A.10})$$

which implies

$$A = A(X - c_g T, \tau) = A(\xi, \tau) \quad (\text{A.11})$$

with $\xi = X - c_g T$, where c_g is the group velocity given by

$$c_g = -2\alpha^2k(2n+1). \quad (\text{A.12})$$

Thus (A.3) can be rewritten in the form

$$yG'' + G' - [k^2y - k(2n+1)]G = (-2ky + 2n+1)\psi, \quad (\text{A.13})$$

where we have chosen $C(\xi, \tau) = iA\xi$. This equation can be solved using variation of parameters giving

$$G(y; \xi, \tau) = \exp(-ky) \left[yL_n(2ky) + \frac{n}{k} L_{n-1}(2ky) \right]. \quad (\text{A.14})$$

To solve (A.4) we first substitute $\psi(y) = \exp(-ky) L_n(2ky)$ yielding

$$\begin{aligned} yH'' + H' - [4k^2y - k(2n+1)]H &= -\frac{2k^2}{\alpha} \exp(-2ky) \\ &\times [3L_n(2ky)L_{n-1}^1(2ky) + [L_{n-1}^1(2ky)]^2 + 2L_n(2ky)L_{n-2}^2(2ky)]. \end{aligned} \quad (\text{A.15})$$

The exponential on the right-hand side of this equation suggests the substitution

$$H(y) = \frac{k}{\alpha} \exp(-2ky) \Psi(y) \quad (\text{A.16})$$

giving

$$z\Psi'' + (1-2z)\Psi' + \left(\frac{2n+1}{2} \right) \Psi = -[3L_nL_{n-1}^1 + (L_{n-1}^1)^2 + 2L_nL_{n-2}^2], \quad (\text{A.17})$$

where we have introduced the change of variable $z = 2ky$ and primes now denote differentiation with respect to z .

For a specific value of n the right-hand side of (A.17) is a polynomial of degree $(2n-1)$. It follows that we can construct a solution in the form

$$\Psi_n(z) = \sum_{k=0}^{2n-1} a_k z^k, \quad (\text{A.18})$$

where the a_k coefficients are found by direct substitution into (A.17). In Table I we present the first few values for these coefficients as a function of n as found using *Mathematica*. Thus, in summary, the solution to (A.4) is given by

$$H(y) = \frac{k}{\alpha} \exp(-2ky) \Psi_n(2ky). \quad (\text{A.19})$$

To facilitate the solution of the $O(\varepsilon^2)$ problem it is convenient to write $h^{(1)}$ in the form

$$h^{(1)} = \frac{iA\xi}{2k} \zeta_1(y) \exp(i\theta) + \frac{A^2 k}{\alpha} \zeta_2(y) \exp(2i\theta) + c.c. + b(y; \xi, \tau), \quad (\text{A.20})$$

where

$$\left. \begin{aligned} \zeta_1(y) &= \exp(-ky) [2kyL_n(2ky) + 2nL_{n-1}(2ky)], \\ \zeta_2(y) &= \exp(-2ky) \Psi_n(2ky). \end{aligned} \right\} \quad (\text{A.21})$$

APPENDIX B

If $h^{(0)}$ and $h^{(1)}$ are substituted into the $O(\varepsilon^2)$ problem, the right-hand side of (4.21) consists of terms proportional to $\exp(0)$, $\exp(\pm i\theta)$, $\exp(\pm 2i\theta)$, and $\exp(\pm 3i\theta)$, respectively.

In order to eliminate the secular behavior associated with the terms independent of the fast phase we demand, after some algebra (see Slomp, 1995), that

$$zb_{zz} + b_z + (2n+1)b = \frac{k}{2\alpha} |A|^2 [4(2n+1)\psi\psi' - 4(\psi')^2 - 8\psi\psi'' + \psi^2]. \quad (\text{B.1})$$

A solution associated with those terms proportional to $\exp(\pm i\theta)$ can, in principle, be found in the form

$$h_p^{(2)} = N(X, T, \tau) \Gamma(y) \exp(i\theta) + c.c.. \quad (\text{B.2})$$

This substitution results in an ordinary differential equation for $\Gamma(y)$ which contains terms involving the mean flow. It is therefore convenient to first solve for the mean flow.

Observing that the homogeneous problem associated with (B.1) is simply Bessel's equation of order zero, allows us to write a Green's function solution to (B.1) in the form

$$\begin{aligned} b(z) &= \pi Y_0 \left[2\sqrt{(2n+1)z} \right] \int_0^z J_0 \left[2\sqrt{(2n+1)\gamma} \right] f(\gamma; \xi, \tau) d\gamma \\ &\quad + \pi J_0 \left[2\sqrt{(2n+1)z} \right] \int_z^\infty Y_0 \left[2\sqrt{(2n+1)\gamma} \right] f(\gamma; \xi, \tau) d\gamma, \quad (\text{B.3}) \end{aligned}$$

where

$$\begin{aligned}
 f(z; \xi, \tau) = & -\frac{k}{2\alpha} |A|^2 \exp(-z) [4(n+1)[L_n(z)]^2 + 8(n+2)L_n(z)L_{n-1}^1(z)] \\
 & -\frac{k}{2\alpha} |A|^2 \exp(-z) [4[L_{n-1}^1(z)]^2 + 8L_n(z)L_{n-2}^2(z)]. \quad (\text{B.4})
 \end{aligned}$$

Because of the dependence of this solution on the Laguerre polynomials and thus on n , it is convenient for further calculations to write the mean flow in the form

$$b(y; \xi, \tau) = \frac{k}{2\alpha} |A|^2 b_n(y).$$

After considerable algebra (see Slomp, 1995), it follows that the ordinary differential equation for Γ can be written in the form (with the change of variable $z = 2ky$)

$$\begin{aligned}
 & N \left[z\Gamma'' + \Gamma + \left(\frac{2n+1}{2} - \frac{z}{4} \right) \Gamma \right] \\
 = & \frac{1}{k^2} A_{\xi\xi} \left[-\frac{2n+1}{2} \zeta_1 + \frac{1}{2} z\zeta_1 - \frac{1}{2} z\psi - (2n+1)\psi \right] \phi \\
 & - \frac{A}{2i\alpha^2 k^2} \psi + \frac{k^2}{2\alpha^2} A|A|^2 (3\psi\zeta_2 - 4\psi'\zeta_2' - 4\psi\zeta_2'' - 4\psi''\zeta_2) \\
 & + \frac{k^2}{2\alpha^2} A|A|^2 \left[8(2n+1)\psi(\psi')^2 + 8(2n+1)\psi^2\psi'' + \frac{\psi b_n}{2} \right] \\
 & + \frac{k^2}{2\alpha^2} A|A|^2 [-2\psi'(b_n)_z(2n+1)\psi(b_n)_z - 2\psi''b_n - 2\psi(b_n)_{zz}]. \quad (\text{B.5})
 \end{aligned}$$

Observing that a homogeneous solution to (B.5) is given by

$$\Gamma_h(z) = \psi(z) = \exp(-z/2)L_n(z)$$

implies, as a consequence of the Fredholm Alternative Theorem, that

$$\begin{aligned}
 & \frac{1}{2k^2} A_{\xi\xi} \int_0^\infty \exp(-z) \left\{ \left[\frac{1}{2}z^2 - (n+1)z - (2n+1) \right] [L_n(z)]^2 \right\} dz \\
 & \quad \frac{1}{2k^2} A_{\xi\xi} \int_0^\infty \exp(-z) [nz - n(2n+1)] L_n(z)L_{n-1}(z) dz
 \end{aligned}$$

$$\begin{aligned}
& -\frac{A_\tau}{2i\alpha^2 k^2} \int_0^\infty \exp(-z)[L_n(z)]^2 dz \\
& + \frac{k^2}{2\alpha^2} A|A|^2 \left[\int_0^\infty (3\psi\zeta_2 - 4\psi'\zeta_2' - 4\psi\zeta_2'' - 4\psi''\zeta_2)\psi dz \right] \\
& + \frac{k^2}{2\alpha^2} A|A|^2 \left\{ 8(2n+1) \int_0^\infty [\psi^2(\psi')^2 + \psi^3\psi''] dz \right\} \\
& + \frac{k^2}{2\alpha^2} A|A|^2 \int_0^\infty \left[\frac{1}{2}\psi b_n - 2\psi'(b_n)_z - (2n+1)\psi(b_n)_z \right] \psi dz \\
& + \frac{k^2}{2\alpha^2} A|A|^2 \int_0^\infty [-2\psi'' b_n - 2\psi(b_n)_{zz}] \psi dz = 0 \tag{B.6}
\end{aligned}$$

must hold.

This integral can be simplified somewhat using integration by parts and the orthogonality condition for Laguerre polynomials. The resulting integrals involving the series solution ζ_2 or Laguerre polynomials were then evaluated using *Mathematica* for each successive value of n . The remaining integrals involving the mean flow, $b(y;\xi,\tau)$, were numerically evaluated. After these calculations are completed (see Slomp, 1995), it follows that (B.6) can be rearranged into

$$A_\tau = \frac{i\omega''}{2} A_{\xi\xi} + ik^4 \beta_n |A|^2 A,$$

where the nonlinear coefficient β_n is tabulated in Table II.