# THE ROLE OF NEGATIVE ENERGY WAVES IN LINEAR AND NONLINEAR SHEAR-FLOW INSTABILITY IN HYPERELASTIC FLUID-FILLED TUBES

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SUMMARY

The linear and nonlinear transition to instability for a shear flow of a density-stratified fluid in a hyperelastic membranous tube is examined. Within the context of the linear stability problem, it is shown that the onset of instability occurs due to the coalescing of positive and negative energy wave modes, where the wave or disturbance energy is defined to be the difference between the phase-averaged energy of the flow in the presence of the wave and the phase-averaged energy of the steady flow. It is shown that even if the linear problem predicts neutral stability, weakly-nonlinear wave–wave interactions can occur between the stable modes to produce explosive instability in finite time. The process of nonlinear destablization is analyzed using the wave activities associated with the disturbance energies.

## 1. Introduction

THE study of wave propagation in compliant fluid-filled tubes is of interest particularly with regard to, among others, blood flow in blood vessels and application to various communications and transmission devices. One of the outstanding problems in this context is the transition to instability or turbulence that can be observed in such a configuration. In particular, it is of interest to investigate the interaction of the dynamics associated with classical shear-flow instability in the fluid, and the dynamics of a deformable tube wall. The principal purpose of this paper is to examine the linear and weakly-nonlinear instability associated with a shear flow in a thin-walled hyperelastic tube containing a density-stratified fluid.

In order to study the nonlinear stability problem, a rational model is required that can describe finite-amplitude deformations of cylindrical elastic shells. The model used here is based on a hyperelastic theory first presented by Moodie and Haddow (1) in their study of shock formation in fluid-filled tubes. Cowley (2, 3) subsequently used this model to investigate the turbulent dissipation of propagating finite-amplitude elastic jumps for this configuration. Swaters (4) also used this model to demonstrate the possibility of the existence of wave-triad interactions in fluid-filled hyperelastic tubes. In this paper, we extend these important studies by including a possible shear flow in the inviscid fluid contained within the tube. There are many flow configurations that can be examined in this context, but for our purpose a simple cylindrical vortex sheet is assumed. It is important to explicitly point out that this is not a terribly realistic model for real flows through a flexible tube. Important physical processes such as fluid viscosity, wall viscoelasticity and the presence of critical layers within the fluid are ignored. Nevertheless, all of the essential features associated with the modulated instability problem can easily be demonstrated, and the calculations associated with the nonlinear stability problem can be explicitly carried out. Another study which is relevant to that present here is the work of Thomas and Craik (**5**), in which the flow of an inviscid fluid over a flexible boundary was considered.

The plan of the paper is as follows. In section 2, the notation and governing equations are introduced. Asymptotic solutions are proposed as power series in a small positive amplitude parameter $\varepsilon$, and the method of multiple scales is used to determine the evolution of the wave amplitudes. In section 3, the linear problem is solved and the dispersion relation, giving the allowable frequencies as a function of wavenumber, is derived. While a general stability criterion is made difficult by algebraic complexities, several special cases are examined, which are used to demonstrate the various aspects of the theory. Linear instability of two types is shown to exist, and is explained in terms of the wave energy. Much of our interpretation of the linear and nonlinear instability problem in terms of negative and positive energy waves is based on ideas presented by Craik (**6**).

Section 4 deals with the solutions of the weakly-nonlinear problem. Here, it is shown that triads of waves can interact in an unstable manner, the occurrence of this being dependent upon the energies of the waves in the triad. It is of significance that this nonlinear effect can be predicted on the basis of information that is obtained from the solution of the linear problem. Finally, section 5 contains a brief summary of the main results, and suggests some areas of further study.

## 2. The governing equations

The fluid is assumed to be confined within a homogeneous, membranous elastic tube, for which longitudinal motion is prevented by the application of tethering forces necessary to prohibit such motion. The properties of axially-tethered, membranous tubes form an area of study on their own (**2**, **3**), and will not be discussed here. We shall, however, make use of some results of nonlinear elasticity theory in assuming that the tube material is hyperelastic, and thus the motion of the tube wall may be expressed in terms of a strain-energy functional $W(\lambda_1, \lambda_2)$, where $\lambda_1$ and $\lambda_2$ are the principal stretches in the azimuthal and longitudinal directions, respectively. Lastly, the inertia of the tube wall is assumed negligible compared to that of the fluid; Cowley (**2**) has shown that this approximation is valid provided that

$$\frac{\rho_m H}{\rho a_0} \ll 1,$$

where $\rho$, $\rho_m$, $a_0$, and $H$ are the fluid and wall densities, and the undeformed wall radius and thickness, respectively.

The fluid itself is assumed to be invisicid, incompressible, and discontinuously stratified, having constant densities within and without an interface specified by an initially constant radius. Only axisymmetric disturbances will be considered here, although the ensuing analysis may be carried out for the general non-axisymmetric problem. The *dimensional* Euler equations for the above problem are

$$(ru)_x + (rv)_r = 0, \tag{2.1}$$

$$u_t + uu_x + vu_r + \rho^{-1}p_x = 0, \tag{2.2}$$

$$v_t + uv_x + vv_r + \rho^{-1}p_r = 0. \tag{2.3}$$

In (2.1), (2.2), and (2.3), $x$, $r$, $t$, $u$, $v$, and $p$ are the dimensional longitudinal and radial coordinates, time, longitudinal and radial velocities, and fluid pressure, respectively, and $\rho$ is the density of the fluid. The appearance of $x$, $t$, or $r$ as a subscript denotes partial differentiation with respect to that variable.

The conditions on the elastic boundary formed by the tube wall can be written in the form (**4**)

$$p = \pi(x, t), \tag{2.4}$$

$$v = a_t + ua_x, \tag{2.5}$$

on $r = a_0 + a(x, t)$, where $a(x, t)$ is the radial deviation of the wall from its undistorted position at $r = a_0$, and $\pi(x, t)$ is the pressure drop associated with the wall elasticity. For the cylindrical hyperelastic tube satisfying the conditions stated earlier, $\pi$ may be written in the form (**1**, **2**, **4**)

$$\pi(x, t) = \frac{H}{(a_0 + a)(1 + e)} \frac{\partial W}{\partial \lambda_1} - \frac{H}{(a_0 + a)} \frac{\partial}{\partial x}\left(\frac{a_0 a_x}{(1 + a_x^2)^{\frac{1}{2}}} \frac{\partial W}{\partial \lambda_2}\right). \tag{2.6}$$

The *dimensional* strain-energy functional is $W(\lambda_1, \lambda_2)$, $e$ is the uniform axial pre-strain, and the azimuthal and longitudinal stretches are given by

$$\lambda_1 = (a_0 + a)/a_0, \tag{2.7a}$$

$$\lambda_2 = (1 + e)(1 + a_x^2)^{\frac{1}{2}}. \tag{2.7b}$$

We remind the reader that the elastic-wall model (2.6) is *not* a linear one. The dependence of the strain-energy functional $W(\lambda_1, \lambda_2)$ on the stretches does not assume small-amplitude deformations of the tube wall; see the discussion in Ogden (**7**).

The above system is solved in each homogeneous region separately, and the solutions are matched across the interface so that the kinematic condition

$$v = \eta_t + u\eta_x \tag{2.8}$$

holds on $r = r_0 + \eta(x, t)$, where $r_0$ is the undisturbed position of the interface, and $\eta(x, t)$ is the dynamic deviation. The dynamic boundary condition on the internal interface (including the effect of surface tension) may be written in the form (**8**)

$$p_o - p_i = \gamma(\eta_{xx}[1 + (\eta_x)^2]^{-\frac{3}{2}} - [r(1 + (\eta_x)^2)^{\frac{1}{2}}]^{-1}), \tag{2.9}$$

evaluated at $r = r_0 + \eta(x, t)$, where $p_o$ and $p_i$ are the limiting values of the fluid pressure on the 'outer' and 'inner' side of the interface $r = r_0 + \eta(x, t)$, respectively, and $\gamma \geqslant 0$ is the surface-tension parameter.

Further analysis is facilitated by the following non-dimensional (asterisked) quantities:

$$\left.\begin{aligned}
&(x, r, a, \eta) = a_0(x^*, r^*, a^*, \eta^*),\\
&(u, v) = c_0(u^*, v^*),\\
&t = (a_0/c_0)t^*,\\
&p = \rho_0 c_0^2 p^*,\\
&W = W_0 W^*,
\end{aligned}\right\} \tag{2.10a}$$

where

$$\left.\begin{aligned}
&c_0^2 = (H/a_0\rho_0)W_0,\\
&x = (\rho_0 c_0^2 a_0)\gamma^*,\\
&r_0 = a_0 r_0^*,
\end{aligned}\right\} \tag{2.10b}$$

and $\rho_0$ is a constant reference density. From this point forward, we shall assume that the non-dimensionalization specified by (2.10) has been carried out, and shall omit asterisks in the equations that result.

It is trivial to verify that the vortex sheet is an exact solution of the fully nonlinear system consisting of the non-dimensional analogues of (2.1) to (2.5), (2.8), and (2.9):

$$u_0 = \begin{cases} U, & r < r_0,\\ 0, & r > r_0;\end{cases} \quad v_0 = 0, \quad a_0 = 0; \quad \eta_0 = 0; \quad p_0 = \begin{cases} P + \gamma/r_0, & r < r_0,\\ P, & r > r_0,\end{cases} \tag{2.11}$$

where

$$P = \frac{1}{1 + e}\frac{\partial W(1, 1 + e)}{\partial \lambda_1},$$

and $U$ is a constant. Since the initial flow is irrotational in each homogeneous region, so is any subsequent flow, and thus we may introduce a velocity potential $\phi(r, x, t)$ such that $\mathbf{u} = \nabla\phi$. Substitution of the velocity potential into (2.2) and (2.3) leads to the Bernoulli function

$$p = -\rho(\phi_t + \tfrac{1}{2}(\nabla\phi)^2), \tag{2.12}$$

modulo an arbitrary function of time. In order to study the stability problem, we introduce the steady and primed anomaly fields in the form

$$\left.\begin{aligned} &\phi = u_0 x + \varepsilon\phi', \quad \mathbf{u} = \nabla\phi, \\ &(a, \eta) = \varepsilon(a', \eta'), \\ &p = p_0 + \varepsilon p', \end{aligned}\right\} \tag{2.13a}$$

where $0 < \varepsilon \ll 1$ is a non-dimensional parameter characteristic of the amplitudes of the disturbances. Substitution of (2.13a) into the nonlinear equations, and the boundary and interfacial conditions given earlier, will show that nonlinearity becomes significant over a space–time scale of $O(\varepsilon^{-1})$. To account for these long time-scale modulations, the slow time (**9**) $T = \varepsilon t$ is introduced; temporal derivatives will therefore transform as

$$\partial_t \mapsto \partial_t + \varepsilon\partial_T, \tag{2.13b}$$

with $t$ and $T$ henceforth treated as independent variables. Following Swaters (**4**), the perturbation fields can be determined by straightforward asymptotic expansions of the form

$$f' \sim \sum_{n=0}^{\infty} \varepsilon^n f^{(n)}(r, x, t; T), \tag{2.13c}$$

$$g' \sim \sum_{n=0}^{\infty} \varepsilon^n g^{(n)}(x, t; T), \tag{2.13d}$$

with $f' \in \{\phi', p'\}$ and $g' \in \{a', \eta'\}$.

Substitution of (2.24) into the non-dimensional equations leads to

$$\nabla^2(\phi^{(0)} + \varepsilon\phi^{(1)}) + O(\varepsilon^2) = 0, \tag{2.14a}$$

$$\begin{aligned} p^{(0)} + \varepsilon p^{(1)} = -\rho\big\{&\phi_t^{(0)} + u_0\phi_x^{(0)} \\ &+ \varepsilon\{\phi_t^{(1)} + u_0\phi_x^{(1)} + \phi_T^{(0)} + \tfrac{1}{2}(\nabla\phi^{(0)})^2\}\big\} + O(\varepsilon^2), \end{aligned} \tag{2.14b}$$

with the Taylor-expanded (about $\varepsilon = 0$) boundary and interfacial conditions:

$$\begin{aligned} \phi_r^{(0)} + \varepsilon\phi_r^{(1)} = \eta_t^{(0)} &+ u_0\eta_x^{(0)} \\ &+ \varepsilon\{\eta_t^{(1)} + u_0\eta_x^{(1)} + \eta_T^{(0)} + \phi_x^{(0)}\eta_x^{(0)} - \eta^{(0)}\phi_{rr}^{(0)}\} \\ &+ O(\varepsilon^2), \quad \text{on } r = r_0, \end{aligned} \tag{2.15a}$$

$$\begin{aligned} [p^{(0)}] + \varepsilon[p^{(1)}] = \gamma\big\{&\eta_{xx}^{(0)} + (\eta^{(0)}/r_0^2) \\ &+ \varepsilon\{\eta_{xx}^{(1)} + (\eta^{(1)}/r_0^2) + (\eta_x^{(0)2}/2r_0) - (\eta^{(0)2}/r_0^3)\}\big\} \\ &- \varepsilon\eta^{(0)}[p_r^{(0)}] + O(\varepsilon^2), \quad \text{on } r = r_0, \end{aligned} \tag{2.15b}$$

$$\phi_r^{(0)} + \varepsilon\phi_r^{(1)} = a_t^{(0)} + \varepsilon\{a_t^{(1)} + a_T^{(0)} + \phi_x^{(0)}a_x^{(0)} - a^{(0)}\phi_{rr}^{(0)}\} + O(\varepsilon^2), \quad \text{on } r = 1, \tag{2.16a}$$

$$\begin{aligned} p^{(0)} + \varepsilon p^{(1)} = {} & (W_{11}^0 - W_1^0)a^{(0)}/(1+e) - W_2^0 a_{xx}^{(0)} \\ & + \varepsilon\big\{(W_{11}^0 - W_1^0)a^{(1)}/(1+e) - W_2^0 a_{xx}^{(1)} \\ & + \big(W_1^0 - W_{11}^0 + (W_{111}^0/2)\big)a^{(0)^2}/(1+e) - (W_{12}^0/2)a_x^{(0)^2} \\ & + (W_2^0 - W_{12}^0)a^{(0)}a_{xx}^{(0)} - a^{(0)}p_r^{(0)}\big\} + O(\varepsilon^2), \quad \text{on } r = 1, \end{aligned} \tag{2.16b}$$

where

$$[f] = f(r_0^+, x, t; T) - f(r_0^-, x, t; T), \tag{2.17}$$

and

$$W_{i_1 i_2 \ldots i_n}^0 = \frac{\partial^n W(1, 1+e)}{\partial\lambda_{i_1}\,\partial\lambda_{i_2} \cdots \partial\lambda_{i_n}}.$$

## 3. The linear stability problem

The $O(1)$-problem is given by

$$\nabla^2\phi^{(0)} = 0, \tag{3.1a}$$

$$p^{(0)} = -\rho(\phi_t^{(0)} + u_0\phi_x^{(0)}), \tag{3.1b}$$

$$\phi_r^{(0)} = \eta_t^{(0)} + u_0\eta_x^{(0)}, \quad \text{on } r = r_0, \tag{3.2a}$$

$$[p^{(0)}] = \gamma\big(\eta_{xx}^{(0)} + (\eta^{(0)}/r_0^2)\big), \quad \text{on } r = r_0, \tag{3.2b}$$

$$\phi_r^{(0)} = a_t^{(0)}, \quad \text{on } r = 1, \tag{3.3a}$$

$$p^{(0)} = \gamma_1 a^{(0)} - \gamma_2 a_{xx}^{(0)}, \quad \text{on } r = 1. \tag{3.3b}$$

The elastic parameters $\gamma_1$ and $\gamma_2$ are defined by

$$\gamma_1 = (W_{11}^0 - W_1^0)/(1+e), \tag{3.4a}$$

$$\gamma_2 = W_2^0, \tag{3.4b}$$

and are known to be positive (**1**, **4**, **7**). Substitution of a travelling-wave solution of the form

$$\phi^{(0)}(r, x, t; T) = \phi(r; T)\exp(ikx - i\omega t), \tag{3.5}$$

where the wavenumber $k$ is assumed to be real, the frequency $\omega$ is allowed to be complex, $i^2 = -1$, and it is understood that only the real part of the right-hand side corresponds to the physical field, into (3.1a) leads to the non-singular general solution of the form

$$\phi^{(0)}(r, x, t; T) = \begin{cases} A(T)I_0(|k|r)\exp(i\theta), & r < r_0, \\ \big(B(T)I_0(|k|r) + C(T)K_0(|k|r)\big)\exp(i\theta), & r > r_0, \end{cases} \tag{3.6}$$

where $\theta = kx - \omega t$ is the phase variable.

The linearity of the $O(1)$-problem implies that $\eta^{(0)}$ and $a^{(0)}$ will be of the form

$$\eta^{(0)}(x, t; T) = A_1(T)\exp(i\theta), \tag{3.7a}$$

$$a^{(0)}(x, t; T) = A_2(T)\exp(i\theta). \tag{3.7b}$$

Conditions (3.2) and (3.3) may then be used to express $A$, $B$, and $C$ as functions of $A_1$ and $A_2$ (see (**10**, Appendix I) for details):

$$A = -i(\omega - Uk)A_1/[|k|I_1(|k|r_0)], \tag{3.8a}$$

$$B = i\omega[A_2K_1(|k|r_0) - A_1K_1(|k|)]/[|k|P(k)], \tag{3.8b}$$

$$C = i\omega[A_2I_1(|k|r_0) - A_1I_1(|k|)]/[|k|P(k)], \tag{3.8c}$$

with

$$P(k) \equiv K_1(|k|)I_1(|k|r_0) - K_1(|k|r_0)I_1(|k|). \tag{3.9}$$

From (3.1b) and (3.2b), it follows that

$$i\rho_2\omega[BI_0(|k|r_0) + CK_0(|k|r_0)] - i\rho_1(\omega - Uk)AI_0(|k|r_0) = \gamma(r_0^{-2} - k^2)A_1,$$

where $\rho_1$ and $\rho_2$ are the densities in $r < r_0$ and $r > r_0$, respectively. With the aid of (3.8), this may be rewritten in the more convenient form

$$D_1(\omega, k)A_1 + \Lambda(\omega, k)A_2 = 0, \tag{3.10}$$

where

$$D_1(\omega, k) \equiv r_0\{\rho_1(\omega - Uk)^2I_0(|k|r_0)/[|k|I_1(|k|r_0)] - \rho_2\omega^2F(k)/[|k|P(k)] + \gamma(r_0^{-2} - k^2)\}, \tag{3.11}$$

$$\Lambda(\omega, k) \equiv \rho_2\omega^2/[k^2P(k)], \tag{3.12}$$

$$F(k) \equiv K_1(|k|)I_0(|k|r_0) + I_1(|k|)K_0(|k|r_0). \tag{3.13}$$

It is not difficult to show that $D_1(\omega, k) = 0$ gives the frequencies $\omega$ as a function of $k$ for waves centred on the undeformed interface when the elastic boundary is replaced by a rigid one; it is thus the dispersion relation for that problem.

Similarly, (3.3b) gives

$$i\rho_2\omega[BI_0(|k|) + CK_0(|k|)] = (\gamma_1 + k^2\gamma_2)A_2.$$

This may be rewritten using (3.8) to obtain

$$D_2(\omega, k)A_2 + \Lambda(\omega, k)A_1 = 0, \tag{3.14}$$

with

$$D_2(w, k) \equiv -\rho_2\omega^2G(k)/[|k|P(k)] - (\gamma_1 + k^2\gamma_2), \tag{3.15}$$

$$G(k) \equiv K_1(|k|r_0)I_0(|k|) + I_1(|k|r_0)K_0(|k|). \tag{3.16}$$

Modes centred on $r = 1$ satisfy $D_2(\omega, k) = 0$ when the interface at $r = r_0$ is replaced by a rigid surface.

Eliminating $A_2$ (or $A_1$) between (3.10) and (3.14), the dispersion relation for the problem at hand is obtained as

$$D(\omega, k) = D_1(\omega, k)D_2(\omega, k) - \Lambda^2(\omega, k) = 0. \tag{3.17}$$

The dispersion relation is thus a quartic in $\omega$, and solutions can be expressed formally as $\omega = \omega(k, U, r_0, \rho_1, \rho_2, \gamma, \gamma_1, \gamma_2)$. Should complex solutions be found to occur, they will occur in complex-conjugate pairs, and so one solution will give rise to an exponentially-growing wave. At this stage, one would generally calculate these roots, and attempt to find conditions under which they are real or complex; this can be done for (3.17) using either solutions by radicals (**11**) or in terms of Weierstrass elliptic functions (**12**). However, the coefficients of $\omega$ are lengthy expressions, and the significance of the various parameters in determining when instability occurs is lost in the algebraic expressions for the roots of $D(\omega, k) = 0$. We shall therefore attempt to understand the physical mechanisms at work by considering a few special cases.

3.1 $U = 0$

In the absence of shear, (3.17) may be written as

$$(ac - e)\omega^4 + (ad + bc)\omega^2 + bd = 0, \tag{3.18}$$

where

$$a \equiv r_0\{\rho_1 I_0(|k|r_0)/[|k|I_1(|k|r_0)] - \rho_2 F(k)/[|k|P(k)]\},$$

$$b \equiv \gamma(r_0^{-2} - k^2), \qquad c \equiv -\rho_2 G(k)/[|k|P(k)],$$

$$d \equiv -(\gamma_1 + k^2\gamma_2), \qquad e \equiv \rho_2^2/[k^4P^2(k)].$$

This is quadratic in $\omega^2$, and the roots are easily found to be

$$\frac{-(ad + bc) \pm ((ad - bc)^2 + 4bde)^{\frac{1}{2}}}{2(ac - e)}. \tag{3.19}$$

It follows that linear instability is predicted whenever the discriminant of (3.19) is negative, or the roots themselves are negative. The discrimimant, regarded as a quadratic in $b$, is found to be always positive, and hence we need only concern ourselves with conditions under which (3.19) is real, but negative. It follows from the monotonic behaviour of the modified Bessel functions that $e - ac < 0$. Thus, linear instability occurs when the numerator of (3.19) is negative; this is so for $|k| < r_0^{-1}$, provided $\gamma \neq 0$ (if $\gamma = 0$, the roots of (3.18) are clearly non-negative). Surface tension thus acts to destabilize the flow for all modes having wavelengths $\lambda = 2\pi/|k|$ which exceed the circumference of the undistorted interface. This is the same stability criterion obtained by Rayleigh (**13**) for the instability of a liquid jet in air; it occurs because surface tension acts to decrease the surface area of the jet (or interface, in the context of the present problem), thereby releasing surface energy.

3.2 *Pipe flow*

When the elastic boundary is replaced by a rigid one, conditions (3.3) are replaced by the condition that the radial component of velocity must vanish on $r = 1$. As mentioned previously, (3.17) reduces to $D_1(\omega, k) = 0$, where $D_1(\omega, k)$ is given by (3.11). The roots are easily computed, and instability is predicted whenever the discriminant is negative, which occurs if either $|k| \leqslant r_0^{-1}$, or $|k| > r_0^{-1}$ and

$$\gamma < \frac{\rho_1 \rho_2 |k| F(k) I_0(|k| r_0) U^2}{[\rho_1 P(k) I_0(|k| r_0) - \rho_2 F(k) I_1(|k| r_0)](r_0^{-2} - k^2)}.$$

Thus, the surface tension acts to suppress the instability for sufficiently short waves.

3.3 *Short-wave approximation*

For those wavenumbers satisfying $|k| r_0 \gg 1$, one may write the terms in (3.17) asymptotically as

$$D_1(\omega, k) D_2(\omega, k) \sim r_0(\rho_1(\omega - Uk)^2 + \rho_2 \omega^2 - \gamma |k|^3)(\rho_2 \omega^2 - \gamma_2 |k|^3)/k^2,$$

$$\Lambda^2(\omega, k) \sim \frac{4 r_0 \rho_2^2 \omega^4}{k^2} \exp(-2|k|(1 - r_0)).$$

The interaction term is thus exponentially subdominant to the other terms, and so

$$D(\omega, k) \sim D_1(\omega, k) D_2(\omega, k).$$

Asymptotically, then, the frequencies are readily verified to be real, provided $\gamma \neq 0$, and complex if $\gamma = 0$.

The assumptions of a homogeneous fluid, whether miscible ($\gamma = 0$) or immiscible does not result in sufficient simplification of (3.17) to easily compute the roots. Similarly, an asymptotic treatment in the case when $|k| r_0 \ll 1$ shows that all terms in (3.17) contribute equally, again resulting in no real simplification.

It is clear that the linear problem is rich and complex; an exhaustive parameter study would be a lengthy procedure. It is fortunate, however, that for the range of parameters considered here, the essential characteristics of the solution to (3.17) appear to be well described by the solutions $D_1(\omega, k) D_2(\omega, k) = 0$, except near points $(\omega, k)$ satisfying both $D_1(\omega, k) = 0$ and $D_2(\omega, k) = 0$. We shall see this as we examine the role of wave energy in predicting instability.

3.4 *Wave energy*

Cairns (**14**) (see also **6**, **15**) showed, in the consideration of a three-layer fluid with step-function velocity and density profiles, that the wave energy, averaged over one wavelength, could be written in the form

$$E = \frac{1}{4}\,\omega \frac{\partial D}{\partial \omega} \times (\text{wave amplitude})^2, \tag{3.20}$$

where $D(\omega, k) = 0$ is the linear dispersion relation. The wave energy in this sense is the phase-averaged difference between the energy of the system when the wave is present, and its energy when the wave is absent. A wave is thus defined to have negative energy if its establishment lowers the wave energy of the system. Further, the Kelvin–Helmholtz (16) instability that occurs in this problem was explained in terms of positive and negative energy waves coexisting on a single interface, and another type of linear instability was shown qualitatively to exist when a positive energy mode propagated on one interface, and a negative energy mode on the other. Craik and Adam (17), in a subsequent paper, verified the correctness of Cairns's prediction; they also carried out a weakly-nonlinear analysis of the same problem, and showed that a type of instability known in plasma physics as 'explosive' instability could take place between triads of waves, with singularities occurring in finite time.

Although it may seem obvious (3.20) also holds in the present context, we shall now confirm, by direct calculation, that this is indeed so. We begin by writing the dispersion relationship in the form

$$D(\omega, k) = D_1(\omega, k) - \frac{\rho_2^2 \omega^4}{k^4 P^2(k) D_2(\omega, k)};$$

upon differentiating the above with respect to $\omega$, and using the fact that $D(\omega, k) = 0$, one obtains

$$E = \frac{1}{4}\left\{-4D_1 + \omega \frac{D_1}{D_2}\frac{\partial D_2}{\partial \omega} + \omega \frac{\partial D_1}{\partial \omega}\right\} |\varepsilon A_1|^2. \tag{3.21}$$

For the fluid interior to the interface at $r = r_0 + \varepsilon\eta$, the contribution to the kinetic energy due to the presence of the $O(1)$-wave is

$$\begin{aligned} KE_{\text{int}} &= \frac{\rho_1}{2}\int_0^{r_0+\varepsilon\eta^{(0)}} r\big((U + \varepsilon u^{(0)})^2 + \varepsilon^2 v^{(0)^2} - U^2\big)\, dr \\ &= \frac{\rho_1 \varepsilon}{2}\int_0^{r_0} r\big(2Uu^{(0)} + \varepsilon(u^{(0)^2} + v^{(0)^2})\big)\, dr \\ &\quad + \frac{\rho_1 r_0 \eta^{(0)} \varepsilon^2}{2}\big(2Uu^{(0)} + \varepsilon(u^{(0)^2} + v^{(0)^2})\big)_{r=r_0} + O(\varepsilon^3), \end{aligned}$$

which, upon substituting $A$ in terms of $A_1$ in the $O(1)$-solutions, averaging over the oscillations, and retaining leading-order terms, gives

$$KE_{\text{int}} = \frac{\rho_1 r_0}{4}\left\{\frac{2kU(\omega - Uk)I_0(|k|r_0) + (\omega - Uk)^2 I_0(|k|r_0)}{|k|I_1(|k|r_0)}\right\}|\varepsilon A_1|^2$$

$$= \frac{1}{4}\left\{\omega\frac{\partial D_1}{\partial \omega} - D_1 + \frac{\rho_2 r_0 \omega^2 F(k)}{|k|P(k)} + r_0\gamma(r_0^{-2} - k^2)\right\}|\varepsilon A_1|^2. \qquad (3.22)$$

By performing a similar calculation for the fluid exterior to the fluid interface, one finds the contribution to the kinetic energy in this region to be, again to $O(\varepsilon^2)$,

$$KE_{\text{ext}} = -\frac{1}{4}\left\{\frac{\rho_2 r_0 \omega^2 F(k)}{|k|P(k)} + \frac{\rho_2 \omega^2 G(k) D_1}{|k|P(k) D_2} + 2D_1\right\}|\varepsilon A_1|^2. \qquad (3.23)$$

Finally, the contribution of the potential energy is (see (**10**, Appendix II))

$$PE = \frac{1}{4}\left((\gamma_1 + k^2\gamma_2)\frac{D_1}{D_2} - r_0\gamma(r_0^{-2} - k^2)\right)|\varepsilon A_1|^2 + O(\varepsilon^3). \qquad (3.24)$$

The total energy is the sum (3.22) + (3.23) + (3.24), and is seen to be equal to the expression given by (3.21). To demonstrate the applications of the energy written in this form, we first consider the behaviour of waves on each interface separately.

Modes centred on $r = 1$ satisfy $D_2(\omega, k) = 0$ when the fluid interface is replaced by a rigid boundary. This configuration simply corresponds to an annular cylinder containing a motionless inviscid fluid with a rigid inner boundary and a hyperelastic outer boundary. Of course we expect these modes to be neutrally stable. The energy of any such wave is readily verified to be

$$E = \frac{1}{4}\omega\frac{\partial D_2}{\partial \omega}|\varepsilon A_2|^2 = -\frac{\rho_2 \omega^2 G(k)}{2|k|P(k)}|\varepsilon A_2|^2 > 0.$$

Thus, on the basis of our energy arguments, these modes do not interact in a destabilizing manner. This is easily seen to be true, since the roots of $D_2(\omega, k) = 0$ are always real.

However, let us suppose that $\omega_1$ and $\omega_2$ are real roots of $D_1(\omega, k) = 0$, for some value of $k$. As pointed out in section 3.2, these are modes which are centred on $r = r_0$ in the case of pipe flow. Writing $D_1(\omega, k) = C_0(\omega - \omega_1)(\omega - \omega_2)$, where $C_0$ is a constant with respect to $\omega$, the energy of these modes is

$$E_1 = \frac{1}{4}\omega_1\left(\frac{\partial D_1}{\partial \omega_1}\right)|\varepsilon A_1|^2 = \tfrac{1}{4}C_0\omega_1(\omega_1 - \omega_2)|\varepsilon A_1|^2,$$

$$E_2 = \frac{1}{4}\omega_2\left(\frac{\partial D_1}{\partial \omega_2}\right)|\varepsilon A_1|^2 = \tfrac{1}{4}C_0\omega_2(\omega_2 - \omega_1)|\varepsilon A_1|^2.$$

Whence, if $\omega_1$ and $\omega_2$ are of the same sign, then their energies are of opposite sign. The discriminant of the solutions of $D_1(\omega, k) = 0$ is found to be a monotonically increasing function of $|k|$, passing through zero (for $\gamma \neq 0$) at some wavenumber $|k| = k_c$. For large $|k|r_0$, the solutions behave asymptotically as

$$\frac{\rho_1 kU \pm |k|^{3/2}(\gamma(\rho_1 + \rho_2))^{1/2}}{\rho_1 + \rho_2}.$$

Provided $U \neq 0$, it follows that there exists $k_0$ such that $\mathrm{sgn}(\omega_1) = \mathrm{sgn}(\omega_2)$ for all $k \in (k_c, k_0)$, and thus, on a graph of $\mathrm{Re}(\omega)$ vs $k$, instability would be predicted near $|k| = k_c$. This instability is demonstrated in Figs 1 and 2 for selected values of the physical parameters, and $\gamma = 5$ and $\gamma = 15$, respectively. The dashed portions of the curves in all of the figures represent unstable modes. It is found that the energy for modes on the lower branch is negative respectively for $k \in [1{\cdot}76, 2{\cdot}04]$ and $k \in [1{\cdot}44, 1{\cdot}52]$, approximately, and positive elsewhere when the solutions are real. Wave energies for modes on the upper branches are always positive in both cases. Note that an increase in $\gamma$ has the effect of stabilizing some modes; however, those modes that remain complex experience an increase in growth rate, given by $|\mathrm{Im}(\omega)|$, with an increase in $\gamma$.

Consider now the complete dispersion relation in the form

$$D(\omega, k) = D_1(\omega, k)D_2(\omega, k) - \rho_2^2\omega^4/k^4P^2(k) = 0,$$

where we shall assume that the modes are only weakly coupled; that is, $|k|r_0$ is not too small. Following the arguments presented in Cairns (14) (see also Craik

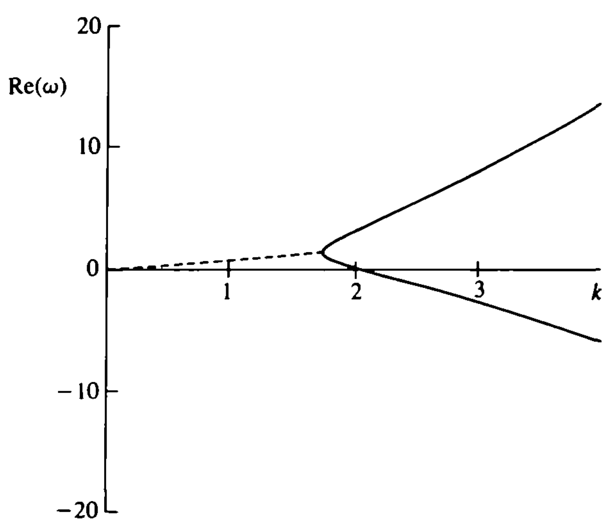


FIG. 1. Dispersion curves for $D_1(\omega, k) = 0$: $U = 2$, $r_0 = 0{\cdot}8$, $\rho_1 = 1{\cdot}02$, $\rho_2 = 1{\cdot}015$, $\gamma = 5$. The dashed portion of the curves corresponds to unstable wavenumbers

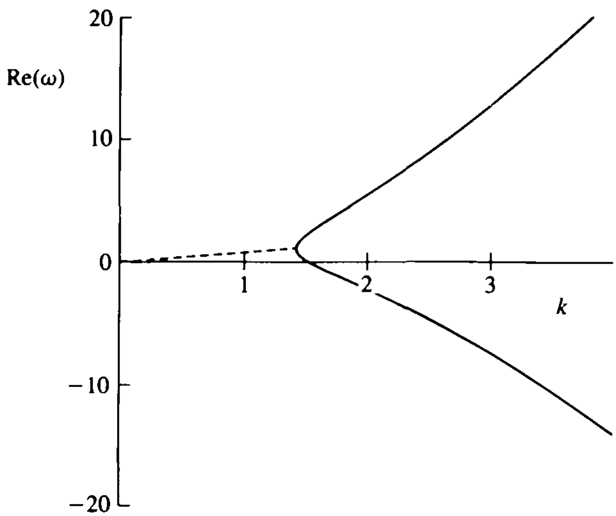


FIG. 2. Dispersion curves for $D_1(\omega, k) = 0$: $U = 2$, $r_0 = 0{\cdot}8$; $\rho_1 = 1{\cdot}02$. $\rho_2 = 1{\cdot}015$, $\gamma = 15$

(6)), let $\omega_1$ and $\omega_2$ be neutrally-stable solutions of $D_1(\omega, k) = 0$ and $D_2(\omega, k) = 0$, respectively, and suppose that these solutions are close:

$$\omega_2 = \omega_1 + \delta, \qquad |\delta/\omega_1| \ll 1.$$

Since the interaction is assumed weak, a solution $\omega$ satisfying $D(\omega, k) = 0$ may be written as

$$\omega = \omega_1 + \Delta, \qquad |\Delta/\omega_1| \ll 1,$$

where, since $\Delta$ may be complex, $|\cdot|$ denotes modulus in the above. Then

$$D_1(\omega, k)D_2(\omega, k) - \rho_2^2\omega^4/k^4P^2(k) = 0,$$

which may be written approximately as

$$\Delta\left(\frac{\partial D_1}{\partial \omega_1}\right)(\Delta - \delta)\left(\frac{\partial D_2}{\partial \omega_2}\right) - \frac{\rho_2^2\omega_1^4}{k^4P^2(k)} = 0.$$

This is a quadratic in $\Delta$, and may be written in the form

$$\Delta^2 - \delta\Delta - T = 0,$$

$$T = \frac{\rho_2^2\omega_1^4}{k^4P^2(k)}\left(\frac{\partial D_1}{\partial \omega_1}\right)^{-1}\left(\frac{\partial D_2}{\partial \omega_2}\right)^{-1} \approx \frac{\rho_2^2\omega_1^6}{k^4P^2(k)}\left(\omega_1\frac{\partial D_1}{\partial \omega_1}\right)^{-1}\left(\omega_2\frac{\partial D_2}{\partial \omega_2}\right)^{-1}.$$

The roots $\Delta$ are easily seen to be real if $T$ is positive, which occurs when the wave energies associated with $\omega_1$ and $\omega_2$ are of the same sign. The roots are complex, implying instability, if $|\delta| < 2|T|^{\frac{1}{2}}$, when $T < 0$. It is to be noted that the instability so predicted arises from the coexistence of waves on different interfaces, but that it can be predicted on the basis of analysing the dispersion properties of waves on each interface separately, under the assumption of weak coupling. This is of great computational value, in particular for multilayered systems, where each interface increases the degree of the dispersion relationship by two.

The onset of this type of instability, known as 'reactive' instability in the parlance of plasma physics, is demonstrated graphically in Figs 3 to 7 for various values of $U$. Note that the ordinate axis begins at $k = 1$; in all cases, the curves increase monotonically from zero in the almost-linear manner that occurs for $k \in [1, 2]$. The interaction term had very little effect on the roots of $D_1(\omega, k) = D_2(\omega, k) = 0$, except near points of intersection of these curves.

In Fig. 3, we see the linear instability for $|k| \leqslant r_0^{-1}$ that is predicted by (3.19). The roots of $D_1(\omega, k) = D_2(\omega, k) = 0$ are relatively well-spaced except near the point where this instability ceases. However, all modes have positive energy near these points, and the only effect of the interaction is to cause an exchange of identities of the roots near the intersection point. For sufficiently large $k$, the minimum and maximum frequencies satisfy $D_1(\omega, k) \approx 0$, while the solutions between these satisfy $D_2(\omega, k) \approx 0$. In Fig. 4, the shear has been increased to unity; there is very little change in the dispersion curves, except for a slight

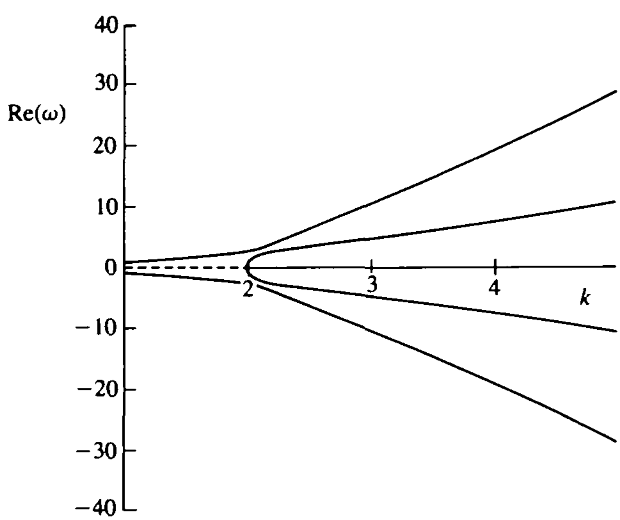


FIG. 3. Dispersion curves for $D_1(\omega, k) = 0$: $U = 0$, $r_0 = 0{\cdot}5$, $\rho_1 = 1{\cdot}02$, $\rho_2 = 1{\cdot}015$, $\gamma = 17$, $\gamma_1 = 1$, $\gamma_2 = 1$

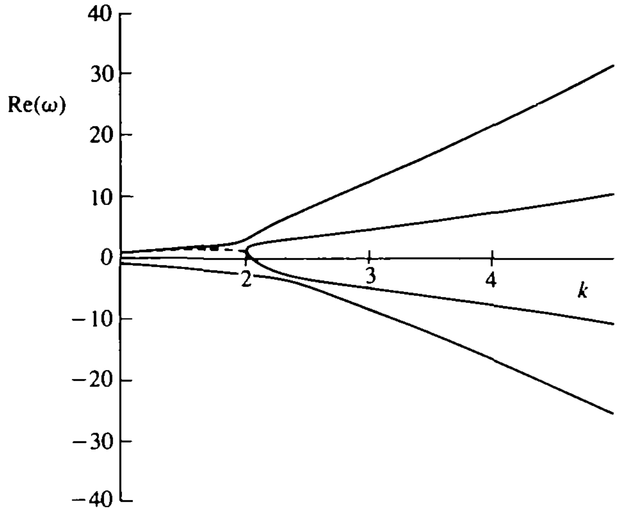


FIG. 4. Dispersion curves for $D_1(\omega, k) = 0$: $U = 1$, $r_0 = 0{\cdot}5$, $\rho_1 = 1{\cdot}02$, $\rho_2 = 1{\cdot}015$, $\gamma = 17$, $\gamma_1 = 1$, $\gamma_2 = 1$

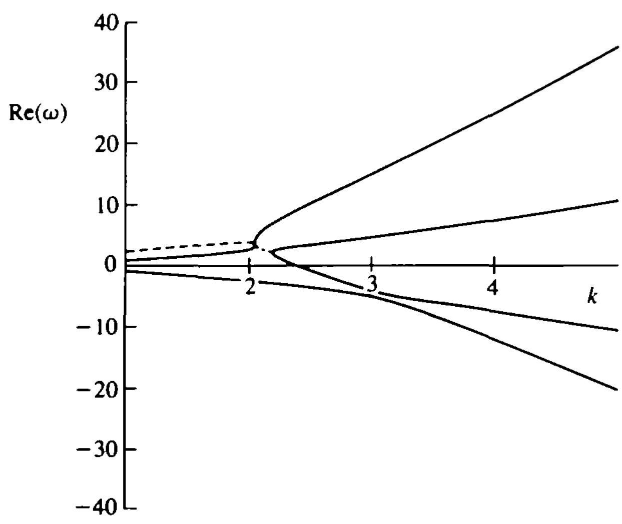


FIG. 5. Dispersion curves for $D_1(\omega, k) = 0$: $U = 2$, $r_0 = 0{\cdot}5$, $\rho_1 = 1{\cdot}02$, $\rho_2 = 1{\cdot}015$, $\gamma = 17$, $\gamma_1 = 1$, $\gamma_2 = 1$

upward displacement of the unstable modes. All neutrally-stable solutions have positive wave energy, except for the lower branch of the unstable modes. These have negative energy in the interval $k \in [2{\cdot}025, 2{\cdot}065]$, and positive energy for $k \geqslant 2.07$, approximately. Thus, the interaction near the intersection point at $k \approx 2{\cdot}4$ is of the passive sort seen previously.

For $U = 2{\cdot}5$, Fig. 5 shows that the real part of the unstable solutions has continued its upward trend, lying now above the other solutions. It is seen that near the first point of intersection at $k \approx 2{\cdot}05$, the interaction between the modes is of the reactive type; the energy of the lower unstable branch is negative between these two regions of instability. It is also negative for $k \in [2{\cdot}195, 2{\cdot}395]$, but this condition is not sustained for sufficiently large $k$ to cause reactive instability near the other intersection point. Figs 6 and 7 show that as shear increases, the region of reactive instability becomes more pronounced, and displays a shift towards larger values of $k$. The behaviour of the solutions for other values of the physical parameters was also investigated (**10**), and it was found that when reactive instability occurred, it appeared in a manner similar to that displayed here.

We remind the reader that our analysis is restricted to axisymmetric modes. In principle, it would be straightforward to incorporate an azimuthal wave-number in the analysis. However, we feel that we have described many of the essential properties of the transition to instability.

We have thus seen that energy considerations may be used to predict the onset of linear instability. In the next section, we shall investigate how these same concepts can be used to study the weakly-nonlinear interactions of the

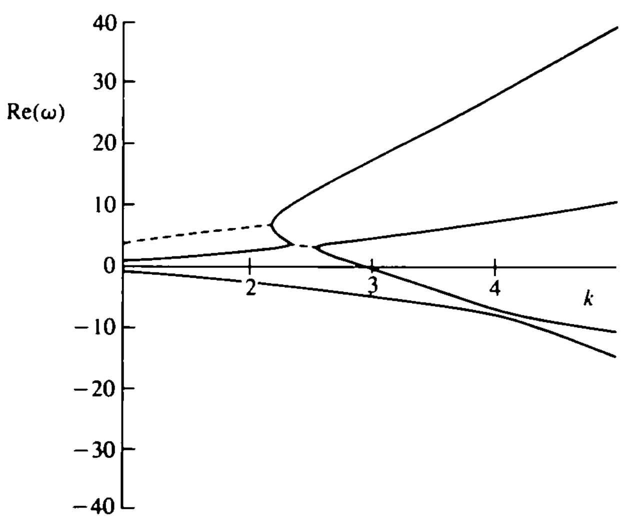


FIG. 6. Dispersion curves for $D_1(\omega, k) = 0$: $U = 4$, $r_0 = 0{\cdot}5$, $\rho_1 = 1{\cdot}02$, $\rho_2 = 1{\cdot}015$, $\gamma = 17$, $\gamma_1 = 1$, $\gamma_2 = 1$

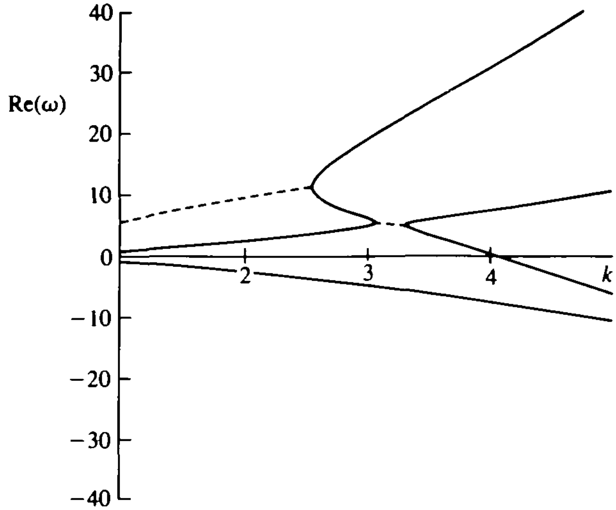


FIG. 7. Dispersion curves for $D_1(\omega, k) = 0$: $U = 6$, $r_0 = 0{\cdot}5$, $\rho_1 = 1{\cdot}02$, $\rho_2 = 1{\cdot}015$, $\gamma = 17$, $\gamma_1 = 1$, $\gamma = 1$

$O(1)$ waves, and predict instability that can occur as a result of these interactions.

## 4. The wave–wave interaction equations

### 4.1 Derivation of the interaction equations

The $O(\varepsilon)$-problem for (2.15) to (2.17) is given by

$$\nabla^2\phi^{(1)}=0, \tag{4.1a}$$

$$p^{(1)}=-\rho(\phi_t^{(1)}+u_0\phi_x^{(1)}+\phi_T^{(0)}+\tfrac{1}{2}(\phi_x^{(0)2}+\phi_r^{(0)2})), \tag{4.1b}$$

$$\phi_r^{(1)}=\eta_t^{(1)}+u_0\eta_x^{(1)}+\eta_T^{(0)}+\phi_x^{(0)}\eta_x^{(0)}-\eta^{(0)}\phi_{rr}^{(0)}, \quad \text{on } r=r_0, \tag{4.2a}$$

$$[p^{(1)}]=\gamma(\eta_{xx}^{(1)}+(\eta^{(1)}/r_0^2)+(\eta_x^{(0)2}/2r_0)-(\eta^{(0)2}/r_0^3))-\eta^{(0)}[p_r^{(0)}], \quad \text{on } r=r_0, \tag{4.2b}$$

$$\phi_r^{(1)}=a_t^{(1)}+a_T^{(0)}+\phi_x^{(0)}a_x^{(0)}-a^{(0)}\phi_{rr}^{(0)}, \quad \text{on } r=1, \tag{4.3a}$$

$$p^{(1)}=\gamma_1a^{(1)}-\gamma_2a_{xx}^{(1)}+\gamma_3a^{(0)2}+\gamma_4a^{(0)}a_{xx}^{(0)}+\gamma_5a_x^{(0)2}-a^{(0)}p_r^{(0)}, \quad \text{on } r=1, \tag{4.3b}$$

where

$$\gamma_3 \equiv (W_1^0 - W_{11}^0 + W_{111}^0/2)/(1+e),$$

$$\gamma_4 \equiv W_2^0 - W_{12}^0, \qquad \gamma_5 \equiv -W_{12}^0/2,$$

and $[p^{(1)}]$ and $[p_r^{(0)}]$ are the pressure jumps as given by (2.17). To study the interaction of resonant triads of waves, the $O(1)$-fields will be written as the superposition of three neutrally-stable waves of the form given by (3.7). Thus

$$\eta^{(0)} = \sum_{n=1}^{3} A_1^{(n)} \exp(i\theta_n), \qquad \theta_n = k_n x - \omega_n t, \tag{4.4}$$

with analogous expressions for the other linear fields. The relationships between the various coefficients, as given by (3.8), are still valid for each $n$. Such a triad is said to be in resonance if any of the following conditions hold:

$$\pm\theta_1 \pm \theta_2 \pm \theta_3 = 0.$$

It will be assumed that the waves in (4.4) satisfy such a condition. Owing to the symmetry of the above in (4.1) to (4.3), we may assume, without loss of generality, that

$$\theta_1 = \theta_2 + \theta_3, \tag{4.5}$$

that is,

$$k_1 = k_2 + k_3, \qquad \omega_1 = \omega_2 + \omega_3.$$

The analysis at this point is virtually identical to that carried out in the linear problem. Earlier works (for example, Simmons (**20**); see also Thomas and Craik (**5**)) suggest that a variational formulation is preferable to the direct method employed here, as the algebraic manipulations involved in the latter are quite formidable. Details of these calculations may be found in (**10**, Appendix III), although the main results are summarized in the Appendix at the end of this paper. The end results are the three-wave interactions, given by

$$-i\frac{\partial D(1)}{\partial \omega_1} A_{1T}^{(1)} = \lambda A_1^{(2)} A_1^{(3)}, \tag{4.6a}$$

$$-i\frac{\partial D(2)}{\partial \omega_2} A_{1T}^{(2)} = \lambda A_1^{(1)} A_1^{(3)*}, \tag{4.6b}$$

$$-i\frac{\partial D(3)}{\partial \omega_3} A_{1T}^{(3)} = \lambda A_1^{(1)} A_1^{(2)*}, \tag{4.6c}$$

where the interaction coefficient $\lambda$ is identical in each equation and is determined in the Appendix.

Conservation of energy follows immediately from these equations; since

$$\begin{aligned}\left(\omega_1 \frac{\partial D(1)}{\partial \omega_1} |A_1^{(1)}|^2\right)_T &= \omega_1 \frac{\partial D(1)}{\partial \omega_1}\left(A_1^{(1)} A_{1T}^{(1)*} + A_1^{(1)*} A_{1T}^{(1)}\right)\\ &= -2\omega_1 \operatorname{Re}(i\lambda A_1^{(1)} A_1^{(2)*} A_1^{(3)*}),\end{aligned}$$

with similar expressions holding for $A_1^{(2)}$ and $A_1^{(3)}$, it follows that

$$\left(\sum_{i=1}^{3} \omega_i \frac{\partial D(i)}{\partial \omega_i} |A_1^{(i)}|^2\right)_T = -2(\omega_1 - \omega_2 - \omega_3) \operatorname{Re}(i\lambda A_1^{(1)} A_1^{(2)*} A_1^{(3)*}).$$

By virtue of the resonance conditions (4.5),

$$(E_1 + E_2 + E_3)_T = 0. \tag{4.7}$$

It is clear from (4.7) that if the signs of the wave energies are the same, the wave amplitudes cannot grow simultaneously. Conversely, should one of the waves have energy of different sign from the other two, it is consistent with energy conservation that the amplitude of this mode could increase/decrease without bound, while the other two amplitudes grow in the opposite sense.

### 4.2 *Explosive instabilities*

Since it is well known how to obtain the general solutions of the three-wave interaction equations (for example, Weiland and Wilhelmsson (**15**), Coppi *et al.* (**18**)), we shall be relatively concise in our presentation here.

If one substitutes

$$A_1^{(1)}(T) = \frac{b_1(T)}{|\lambda_2\lambda_3|^{\frac{1}{2}}}\exp(i\psi_1(T)),$$

$$A_1^{(2)}(T) = \frac{b_2(T)}{|\lambda_1\lambda_3|^{\frac{1}{2}}}\exp(i\psi_2(T)),$$

$$A_1^{(3)}(T) = \frac{b_3(T)}{|\lambda_1\lambda_2|^{\frac{1}{2}}}\exp(i\psi_3(T)),$$

where $b_j(T)$ and $\psi_j(\mathrm{T})$ are real-valued, and

$$\lambda_j = -\lambda\left(\frac{\partial D(j)}{\partial \omega_j}\right)^{-1}$$

into (4.6), it follows that

$$b_{1_T} = -s_1 b_2 b_3 \sin\psi, \quad b_{2_T} = s_2 b_1 b_3 \sin\psi, \quad b_{3_T} = s_3 b_1 b_2 \sin\psi, \tag{4.8}$$

$$\psi_T = \left(-\frac{s_1 b_2 b_3}{b_1} + \frac{s_2 b_1 b_3}{b_2} + \frac{s_3 b_1 b_2}{b_3}\right)\cos\psi, \tag{4.9}$$

with $s_j = \mathrm{sgn}(\lambda_j)$ and $\psi = \psi_1 - \psi_2 - \psi_3$. It follows readily from (4.8) that we have the following constants of motion, the Manley–Rowe relations:

$$s_1 b_1^2(T) + s_2 b_2^2(T) = M_1, \qquad s_1 b_1^2(T) + s_3 b_3^2(T) = M_2, \tag{4.10a}$$

where $M_1$ and $M_2$ represent the values of the left-hand sides at $T = 0$. If (4.9) is multiplied by $\sin\psi$ and one uses (4.8) in the result, one also finds the conserved quantity

$$b_1(T) b_2(T) b_3(T)\cos\psi(T) = \Gamma. \tag{4.10b}$$

The constant $\Gamma$ is characteristic of the strength of the interactions between the resonating waves. By this it is meant that if $\psi(0) = \frac{1}{2}\pi$, then (4.10b) guarantees that $\psi(T) = \frac{1}{2}\pi$ for arbitrary initial amplitudes, and thus the interactions described by (4.8) are at their maximum.

It is clear that if $s_1 = s_2 = s_3$, then (4.10a) guarantees that the $b_j(T)$ remain bounded for all $T \in [0, \infty)$. This situation corresponds to the case in which the signs of the wave energies $E_1$, $E_2$ and $E_3$ are all the same. If, however, $s_2 = s_3 = -s_1 = 1$, the signs of the individual wave energies are not all the same and the triad is explosively unstable (Coppi *et al.* (**18**), Weiland and Wilhelmsson (**15**), and Craik and Adam (**17**)).

To determine whether or not the present system can sustain triads of the above type, one must first determine whether or not triads satisfying (4.5) exist. Since the dispersion relation (3.17) is a quartic, there are four modes for a given $k$, and thus 64 branches to be checked for a given triad satisfying $k_1 = k_2 + k_3$. However, only one of the neutrally-stable branches has negative wave energy, and thus the search is limited by the fact that at least one of the roots must lie on this branch if explosive instability is to occur. Many authors have used an elegant graphical technique (see Phillips (**19**), Simmons (**20**), Craik and Adam (**17**)) for determining the location of resonantly-interacting triads, and this technique was employed here to obtain an estimate of their values. This estimate was made more precise by finding the roots of $\omega(k_1) - \omega(k_2) - \omega(k_3) = 0$ numerically, and the values of $k_2$ and $k_3$ satisfying this condition so obtained are illustrated in Fig. 8. The values of $\omega(k_2)$ and $\omega(k_3)$ lie on the branch labelled

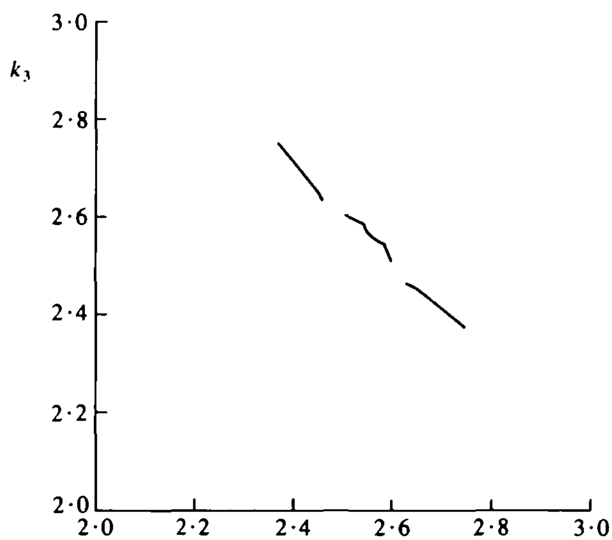


FIG. 8. Resonant Triads ($k_1 = k_2 + k_3$): $U = 6, r_0 = 0{\cdot}5, \rho_1 = 1{\cdot}02, \rho_2 = 1{\cdot}015,$
$\gamma = 50, \gamma_1 = 5, \gamma_2 = 0$

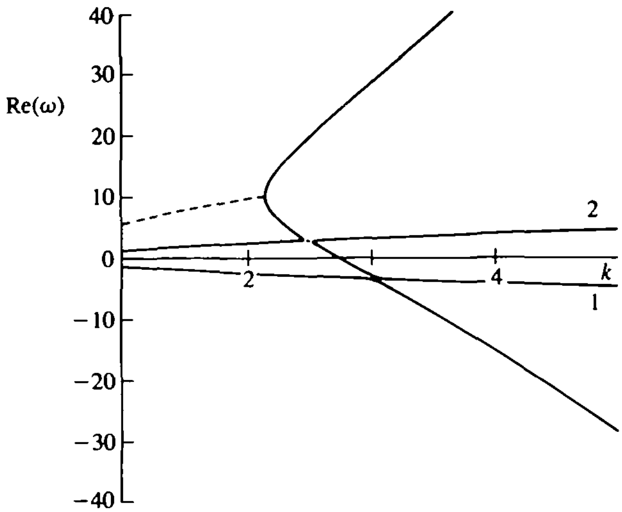


FIG. 9. Dispersion curves $D(\omega, k) = 0$ for resonant triads: $U = 6$, $r_0 = 0{\cdot}5$, $\rho_1 = 1{\cdot}02$, $\rho_2 = 1{\cdot}015$, $\gamma = 50$, $\gamma_1 = 5$, $\gamma_2 = 0$. This branch labelled 2 has positive energy modes and the negative energy modes are located on branch 1

1 in Fig. 9; the energy is negative for $k \in [2{\cdot}175, 2{\cdot}465] \cup [2{\cdot}55, 2{\cdot}75]$, approximately, and positive elsewhere when the solutions are real. The solutions $\omega(k_1)$ lie on the branch labelled 2; all modes on this branch have positive energy, as do those on the remaining two branches. The cutoffs for $k_2$ and $k_3$ in Fig. 8 occur because the solutions are complex for $k \in (2{\cdot}465, 2{\cdot}55)$, approximately. Thus, we have found wavenumber intervals for which resonantly-interacting waves become explosively unstable.

## 5. Conclusion and discussion

The concept of wave energy, as defined in the preceding discussion, has proven useful in the study of the stability properties of some parallel shear flows. In the problem just examined, it was shown that both linear and weakly-nonlinear instabilities could be interpreted in terms of the coexistence of positive- and negative-energy wave modes. In the linear case, the principal effects leading to exponentially-growing solutions, and thus to a breakdown in the validity of the linear approximation, are the kinetic energy due to the presence of the shear flow, and the surface energy released by a decrease in interfacial surface area. The latter is reflected by the simultaneous propagation of modes having wave energies of opposite signs on the fluid interface, while the former is associated with the coexistence of a positive-energy mode on the elastic boundary and a negative-energy mode on the fluid interface. The 'explosive' instabilities found to occur among resonantly-interacting triads of waves are due to the exchange of energy among these modes in a manner consistent with energy conservation. It was demonstrated that for this to occur, one member of the triad must have wave energy of opposite sign to the other two. The wave energy used in all cases was defined in terms of information obtained in the linear analysis.

Viscous effects in the fluid, within the context of a smooth velocity profile, could be studied to provide a more accurate description of the physical system studied here. In addition, it is known that viscoelasticity is a significant factor in the response of the tube wall to elastic deformations (**21**, **22**, **23**), while fluid viscosity is generally a stabilizing effect, since friction may serve to dissipate energy that might otherwise feed an instability. However, this very fact could cause instability of those modes possessing negative wave energy. This has been found to occur in some plane parallel shear flows (**24**, **25**), but it is unknown as yet whether or not this occurs here.

From a more general point of view, the linear treatment of the problem presented here is by no means complete. In the classical theory of plane parallel shear flow, general stability criteria have been derived, giving necessary conditions on the initial velocity and density profiles in order for modal instability to occur. There is a disappointing lack of such results for flows in cylindrical systems, most notably an analogue of Squire's theorem, so that the axisymmetric stability analysis performed here gives no information about the general non-axisymmetric problem. While it is relatively straightforward to prove special cases of such results, the boundary and interfacial conditions in the present problem have, as yet, made the derivation of such theorems impossible in the general case. Further research into this area could contribute significantly to a more comprehensive understanding of fluid flows in cylindrical systems.

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## APPENDIX

*Calculations leading to the three-wave interaction equations*

The linear fields and their $O(\varepsilon)$ counterparts are distinguished by a caret over the amplitude, viz.

$$\eta^{(0)} = \sum_{n=1}^{3} A_1^{(n)} \exp(i\theta_n),$$

$$\eta^{(1)} = \sum_{n=1}^{3} \hat{A}_1^{(n)} \exp(i\theta_n),$$

with $\theta_n = k_n x - \omega_n t$. The $O(\varepsilon)$ amplitudes may be expressed as follows:

$$\hat{A}^{(n)} = \frac{-i(\omega_n - Uk_n)\hat{A}_1^{(n)} + A_{1T}^{(n)} + R(\theta_n)}{|k_n| I_1(|k_n| r_0)}, \tag{A1a}$$

$$\hat{B}^{(n)} = \frac{K_1(|k_n|)[-i\omega_n \hat{A}_1^{(n)} + A_{1T}^{(n)} + S(\theta_n)] - K_1(|k_n||r_0)[-i\omega_n \hat{A}_{2T}^{(n)} + A_{2T}^{(n)} + T(\theta^n)]}{|k_n| P(k_n)}, \tag{A1b}$$

$$\hat{C}^{(n)} = \frac{I_1(|k_n|)[-i\omega_n \hat{A}_1^{(n)} + A_{1T}^{(n)} + S(\theta_n)] - I_1(|k_n| r_0)[-i\omega_n \hat{A}^{(n)} + A_{2T}^{(n)} + T(\theta_n)]}{|k_n| P(k_n)}, \tag{A1c}$$

where

$$R(\theta) \equiv -\eta^{(0)}\phi_{rr}^{(0)}(r_0^-) + \eta_x^{(0)}\phi_x^{(0)}(r_0^-), \tag{A2a}$$

$$S(\theta) \equiv -\eta^{(0)}\phi_{rr}^{(0)}(r_0^+) + \eta_x^{(0)}\phi_x^{(0)}(r_0^+), \tag{A2b}$$

$$T(\theta) \equiv -a^{(0)}\phi_{rr}^{(0)}(r = 1) + a_x^{(0)}\phi_x^{(0)}(r = 1), \tag{A2c}$$

These hold for $n = 1, 2, 3$, with $P(k_n)$ given by (3.9), and $R(\theta_n)$, $S(\theta_n)$, and $T(\theta_n)$ are understood to be those quantities in (A2) that have phase $\theta_n$. Referring again to (10,

Appendix III), application of the pressure condition (4.2b) yields

$$D_2(n)\hat{A}_2^{(n)} + \Lambda(n)\hat{A}_1^{(n)} = \chi(n)A_{1T}^{(n)} + \Omega(n), \tag{A3a}$$

where the notation $f(n) = f(\omega_n, k_n)$ will be used for the remainder of this section. The condition governing the change in pressure across the elastic boundary, (4.3b), gives

$$D_1(n)\hat{A}_1^{(n)} + \Lambda(n)\hat{A}_2^{(n)} = \Phi(n)A_{1T}^{(n)} + \Gamma(n). \tag{A3b}$$

In the above, $D_1$, $D_2$, and $\Lambda$ have their usual meanings from (3.11), (3.15), and (3.12), respectively. The coefficients $\chi(n)$, $\Omega(n)$, $\Phi(n)$, and $\Gamma(n)$ are given by (see (**10**), for details)

$$\Phi(n) \equiv ir_0\left[\frac{\rho_2\omega_n}{|k_n|P(k_n)}\left(F(k_n) - \frac{D_1(n)}{|k_n|r_0\Lambda(n)}\right) - \frac{2\rho_1(\omega_n - Uk_n)I_0(|k_n|r_0)}{|k_n|I_1(|k_n|r_0)} + \frac{\rho_2 D_1(n)}{k_n r_0}\right],$$

$$\Gamma(n) \equiv ir_0\left[\frac{\rho_2\omega_n}{|k_n|P(k_n)}\left(S(\theta_n)F(k_n) - \frac{T(\theta_n)}{|k_n|r_0}\right) - \frac{\rho_1(\omega_n - Uk_n)I_0(|k_n|r_0)R(\theta_n)}{|k_n|I_1(|k_n|r_0)} + iQ(\theta_n)\right],$$

$$\chi(n) \equiv -i\rho_2\left[\frac{\omega_n}{k_n^2 P(k_n)} + \frac{D_1(n)D_2(n)}{k_n\Lambda(n)} + \frac{\omega_n G(k_n)D_1(n)}{|k_n|P(k_n)\Lambda(n)}\right],$$

$$\Omega(n) \equiv -\frac{i\rho_2\omega_n}{|k_n|P(k_n)}\left(\frac{S(\theta_n)}{|k_n|} - T(\theta_n)G(k_n)\right) + P(\theta_n).$$

It will be recalled that the wavenumber/frequency pairs $(k_n, \omega_n)$ are neutrally-stable solutions of $D(\omega, k) = 0$; whence, dividing (A3b) by $D_1(n)$, (A3a) by $\Lambda(n)$, and subtracting, one finds that

$$\big(\Phi(n)\Lambda(n) - D_1(n)\chi(n)\big)A_{1T}^{(n)} = D_1(n)\Omega(n) - \Lambda(n)\Gamma(n),$$

which is the evolution equation for the $n$th $O(1)$ amplitude. It can be shown (**10**) that

$$\Phi(n)\Lambda(n) - D_1(n)\chi(n) = -i\Lambda(n)\frac{\partial D(n)}{\partial\omega_n},$$

while

$$D_1(n)\Omega(n) - \Lambda(n)\Gamma(n) = \Lambda(n)\lambda\beta(n).$$

In the above,

$$\beta(1) = A_1^{(2)}A_1^{(3)}, \quad \beta(2) = A_1^{(1)}A_1^{(3)*}, \quad \beta(3) = A_1^{(1)}A_1^{(2)*},$$

where $*$ denotes complex conjugation. The interaction coefficient $\lambda$ is independent of $n$, and can be shown to be given by (after substantial algebra; see (**10**))

$$\lambda = \tfrac{1}{4}(\mathscr{V}' + \gamma(k_1^2 - k_2k_3)),$$

where

$$\begin{aligned}\mathscr{V}' \equiv \mathscr{V} &- \frac{3\gamma}{r_0^2} + \Theta\left(W_{12}^0(k_2^2 + k_3^2 + k_2k_3) + \frac{W_{111}^0}{1+e}\right)\\ &+ \rho_2\Theta\left(\frac{\omega_1\mathscr{D}_2(1)}{k_1} + \frac{\omega D_2(2)}{k_2} + \frac{\omega_3\mathscr{D}_2(3)}{k_3}\right) - \rho_2(\Theta + r_0)(\omega_2^2 + \omega_3^2 + \omega_2\omega_3)\\ &+ r_0\rho_1((\omega_2 - Uk_2)^2 + (\omega_3 - Uk_3)^2 + (\omega_2 - Uk_2)(\omega_3 - Uk_3)),\end{aligned}$$

$$\mathscr{V} \equiv \frac{\rho_2}{r_0}(\mathscr{D}_1(1)\mathscr{D}_1(2) + \mathscr{D}_1(1)\mathscr{D}_1(3) + \mathscr{D}_1(2)\mathscr{D}_1(3))$$
$$+ \rho_2\Theta(\mathscr{D}_2(1)\mathscr{D}_2(2) + \mathscr{D}_2(1)\mathscr{D}_2(3) + \mathscr{D}_2(2)\mathscr{D}_2(3))$$
$$+ \mathscr{F}(1,2) + \mathscr{F}(1,3) + \mathscr{F}(2,3),$$

and

$$\mathscr{D}_1(n) \equiv \frac{k_n}{\rho_2\omega_n}\left\{D_1(n) + \frac{r_0\rho_0\omega_n^2 F(k_n)}{|k_n|P(k_n)}\right\},$$

$$\mathscr{D}_2(n) \equiv \frac{k_n}{\rho_2\omega_n}\left\{D_2(n) + \frac{\rho_2\omega_n^2 G(k_n)}{|k_n|P(k_n)}\right\},$$

$$\mathscr{F}(i,j) \equiv -\frac{r_0\rho_1 k_i k_j(\omega_i - Uk_i)(\omega_j - Uk_j)I_0(|k_i|r_0)I_0(|k_j|r_0)}{|k_i k_j|I_1(|k_i|r_0)I_1(|k_j|r_0)},$$

$$\Theta \equiv \frac{D_1(1)D_1(2)D_1(3)}{\Lambda(1)\Lambda(2)\Lambda(3)}.$$