On the baroclinic dynamics, hamiltonian formulation and general stability characteristics of density-driven surface currents and fronts over a sloping continental shelf

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A new theory is presented to describe the baroclinic dynamics of density-driven currents and fronts over a sloping continental shelf. The frontal dynamics is geostrophic to leading-order but not quasi-geostrophic since the dynamic frontal height is not small in comparison with the scale frontal thickness. The evolution of the underlying slope water is modelled quasigeostrophically and includes the influence of a background vorticity gradient due to the sloping bottom. The two layers are coupled together via baroclinic vortex-tube stretching associated with the perturbed density-driven current. The current dynamics includes the advection of mean flow vorticity. The model equations are obtained in a formal asymptotic expansion of the relevant two-layer shallow-water equations and boundary conditions. It is shown that the governing equations for the model can be put into non-canonical hamiltonian form.

A comprehensive analysis of the general linear and nonlinear stability characteristics of the governing equations is given. The normal mode problem associated with steady along-shore currents is studied and sufficient stability and necessary instability conditions are presented. It is shown that a zero in the frontal vorticity gradient is not needed for instability. Jump conditions for the perturbation frontal thickness are systematically derived associated with the continuity of pressure and normal mass flux for steady frontal configurations that possess discontinuities in the velocity or vorticity, and rigorous regularity conditions are obtained for the perturbation thickness on outcroppings. The *formal* stability of arbitrary steady currents is studied. It is shown how to obtain general steady current solutions as a variational solution to a suitably constrained hamiltonian. General criteria are obtained for establishing the linear stability of these steady density-driven currents in the sense of Liapunov. In the limit of steady parallel along-shore flow, the formal stability results reduce to the sufficient conditions found for the normal modes. Finally, the nonlinear stability of steady density-driven currents and fronts is studied. Based on the formal stability analysis, appropriate convexity hypothesis are found that rigorously establish nonlinear stability of steady currents in the sense of Liapunov, and establish nonlinear saturation bounds on the perturbation flow with respect to a potential enstrophy/energy norm.

Phil. Trans. R. Soc. Lond. A (1993) **345**, 295–325

Printed in Great Britain

1. Introduction

One of the underlying assumptions in the quasigeostrophic theory for a stratified rotating fluid is the requirement that the vertical deflections of the isopycnals is small in comparison with the scale depth. This means, for example, in the context of a two-layer model that the magnitude of the dynamic deflections of the interface between the two layers is implicitly assumed to be small relative to both of the individual layer mean thicknesses. This assumption leads to significant difficulties in attempting to apply baroclinic quasigeostrophic theory to model fronts and currents in, for example, the coastal regions of the world oceans where it is typically the case that isopycnals associated with these flows intersect the surface or the bottom. Notwithstanding these points, the application of quasigeostrophic theory has been able to explain, at least qualitatively, some aspects of the baroclinic instability observed in some frontal flows (see, for example, Orlanski 1968; Smith 1976; Mysak *et al.* 1981; Griffiths & Linden 1981; Mertz *et al.* 1990). To a large extent, much of the theoretical work on frontal instability done over the last decade is based on Griffiths *et al.* (1982), hereafter referred to as GKS. The dynamical model that GKS developed was based on a semi-geostrophic approximation in which the along-front velocities were geostrophically determined and the across-front or transverse velocities where determined ageostrophically. Baroclinicity was reduced to its simplest form by examining an equivalent-barotropic or reduced-gravity model. Even with all these approximations, the resulting linear stability equations could not be solved analytically except in the zero along-front wavenumber limit for a zero potential vorticity flow. The specific physical problem that GKS examined was the stability of a mesoscale gravity current on a linearly sloping bottom. The stratification characteristics of a flow of this type will in general contain isopycnals that typically intersect the bottom both on the onshore and offshore side of the flow, i.e. a coupled front configuration. In addition to developing an analytical theory for the stability problem, GKS compared the theoretical results against experimental observations. There were, however, substantial differences between the two. In particular, the observed instabilities occurred over a range of finite wavenumbers (including the most unstable mode), while the theory formally required asymptotically small wavenumbers. A second difficulty was that the observed instabilities do not appear to depend sensitively on the current width in contradiction to the theory. These differences were attributed to the presence of another, possibly baroclinic, unstable mode outside the range of applicability of the GKS analysis.

Swaters (1991) developed a baroclinic instability theory for mesoscale gravity currents on a sloping bottom in an attempt to explain these differences. This model differed from the GKS model in two key aspects. Specifically, the flow field in the gravity current was assumed geostrophic (for which there is observational evidence, e.g. LeBlond *et al.* (1991)), and a second assumption was that the gravity current strongly interacts with the surrounding fluid through vortex stretching (as suggested by the Stern theorem for isolated cold-core eddies on a sloping bottom, e.g. Mory *et al.* (1987)). The Swaters' theory was able to describe several of the instability features seen in the GKS experiments. For example, the most unstable mode had a finite non-zero wavelength and the instability characteristics did not depend sensitively on the underlying current width. Another interesting aspect of the GKS experiments was the observation of a dipole 'cyclone–anticyclone' pair instability for some regions of parameter space. The GKS model was unable to describe this mode. The Swaters' model was able to reproduce this second mode for certain parameter values.

Killworth & Stern (1982) applied the equivalent-barotropic GKS theory to *surface* fronts and currents along an oceanic boundary and made a qualitative comparison between the results of their theory and the experimental observations made by Griffiths & Linden (1981, 1982) on density-driven currents. While there was some agreement between the two, there were significant differences particularly with respect to predicted and observed growth rates and wavelengths. Some of these differences may be attributed to the important baroclinic nature of the flow configuration as suggested by Griffiths & Linden (1981). The principal purpose of this paper is to present a theory describing the baroclinic dynamics of buoyancy-driven coastal currents and fronts over a sloping bottom and to provide a detailed mathematical analysis of the hamiltonian structure and general linear and nonlinear stability properties of this model.

The new model presented here focuses in on three kinematical and dynamical processes that are important in the transition to instability and hence eddy formation for coastal flows with isopycnal outcroppings. Specifically, we model the instability process as a dynamical balance between the release of mean kinetic energy from the front, the generation of relative vorticity in the surrounding slope water by baroclinic vortex-tube stretching due to the perturbed front, and the rectifying influence of the underlying background vorticity gradient associated with a sloping bottom. In particular, we will show that without *baroclinic* processes, our model predicts stability. This point is important because it implies that the modes that are described by this model are not simply baroclinically-modified instabilities of the kind obtained in previous models, e.g. Killworth & Stern (1982), but represent a new class of unstable baroclinic modes for buoyancy-driven coastal currents. Our model is derived in a formal asymptotic expansion of the two-layer shallow water equations.

Initially, it was thought that the theory developed by Swaters (1991) for mesoscale gravity currents or coupled fronts on a sloping bottom could be simply 'inverted' to provide a model that could describe the baroclinic instability of the surface density-driven currents that we are interested in here. However, as it turns out, the asymptotic limit examined in Swaters (1991) when applied to surface currents does not yield a dynamically interesting system of equations from the point of view of the instability problem. The underlying reason is that the dynamical processes that lead to baroclinic instability are rather different between mesoscale bottom gravity currents and surface buoyancy-driven currents. The instability described in Swaters (1991) is gravitationally-induced due to the density contrast and the fact that the coupled front is located directly on a sloping bottom leading to an offshore 'slumping'. The instability can therefore be described as convective in nature and thus purely baroclinic. For a surface buoyancy-driven current, the lighter fluid has in some sense nowhere to rise to and thus the instability cannot be convective in nature. Consequently, for the instability to proceed there must be a release of available kinetic energy from the mean flow. However, for this release to occur the advective nonlinear terms in the current momentum equations must be retained in some limit. The Swaters (1991) theory is based on an asymptotic limit which neglects these nonlinear terms entirely. For the present physical situation a new model has to be developed from first principles. The theory that we develop here is similar in some respects to Flierl's (1984) model for warm-core eddies and Cushman-Roisin's (1986) model for surface fronts.

The outline of this paper is as follows. In §2 the physical motivation for our scaling of the two-layer shallow-water equations is presented and the asymptotic expansion is introduced. In particular, a detailed discussion of the boundary conditions on an outcropping is given. As well, in §2, the hamiltonian formulation of the model is presented to set the stage for our subsequent general stability analyses. The general form of the Casimir functionals is derived and we show how to obtain steady flows as the solution to a variational problem for a suitably constrained hamiltonian.

In §3, the *linear* stability problem is studied in detail. We begin by deriving the linear stability equations and boundary conditions and make some qualitative remarks about the energetics of instability. We then turn to an analysis of the normal-mode equations associated with *along-shore* steady flows. We present two stability results for these flows and interpret these stability and instability conditions in terms of the leading-order potential vorticity gradient associated with the density-driven current.

In §4 we turn to presenting a linear and nonlinear stability analyses of steady flows. To begin with we establish conditions for the *formal* stability (Holm *et al.* 1985) and linear stability in the sense of Liapunov based on the non-canonical hamiltonian formulation presented in §2. In the limit of steady along-shore flows, it is shown that the formal stability conditions reduce to the normal-mode stability conditions obtained in §3. After analysing the formal stability of general solutions, appropriate convexity conditions are obtained that can establish the *nonlinear* stability in the sense of Liapunov for general steady density-driven currents and fronts. We also present a nonlinear saturation bound for the growth of disturbances in terms of a potential enstrophy/energy norm on the initial perturbation. The paper is summarized in §5.

2. Derivation of the baroclinic model and hamiltonian structure

(*a*) *Evolution equations*

The basic model we assume is a two-layer shallow-water system (both layers are assumed hydrostatic homogeneous and incompressible) with a linearly sloping bottom as depicted in figure 1. Assuming a rigid-lid, the *dimensional* equations of motion can be written in the form

$$[\partial_{t^*} + \boldsymbol{u}_1^* \cdot \nabla^* + f_0\, \hat{e}_3\, \boldsymbol{x}]\, \boldsymbol{u}_1^* = -g\nabla^* \eta^*, \tag{2.1a}$$

$$h_{t^*}^* + \nabla^* \cdot [\boldsymbol{u}_1^*\, h^*] = 0, \tag{2.1b}$$

for the upper layer (one), and

$$[\partial_{t^*} + \boldsymbol{u}_2^* \cdot \nabla^* + f_0\, \hat{\boldsymbol{e}}_3\, \boldsymbol{x}]\, \boldsymbol{u}_2^* = -\nabla^*(p^*/\rho_2), \tag{2.2a}$$

$$h_{t^*}^* + \nabla^* \cdot [\boldsymbol{u}_2^*(h^* - s^* y^* - H)] = 0, \tag{2.2b}$$

for the lower layer (two), with the reduced pressures η^* and p^* in layers one and two, respectively, related via

$$p^* = g\rho_1\, \eta^* - g(\rho_2 - \rho_1)\, h^*, \tag{2.3}$$

where we are also assuming $\rho_2 > \rho_1$ (stable stratification) where ρ_1 and ρ_2 are the

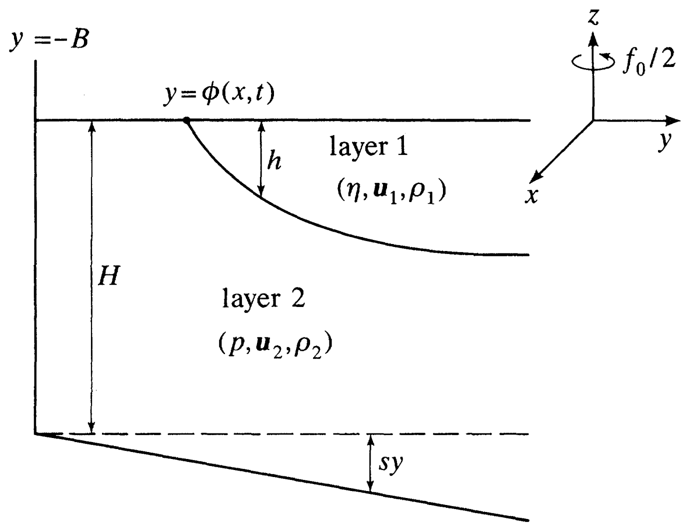

Figure 1. General geometry of the two-layer model used in this paper.

densities of the upper and lower layers respectively. To focus attention on baroclinic processes we have introduced the rigid-lid approximation into (2.1*b*) and (2.2*b*). Subscripts with respect to (x^*, y^*, t^*) indicate partial differentiation and $\nabla^* \equiv (\partial_{x^*}, \partial_{y^*})$. The Coriolis parameter f_0 is assumed constant. The dependent variables h^* and $\boldsymbol{u}_1^*$ correspond to the total depth and eulerian velocity field in layer one respectively. The dependent variables s^* and $\boldsymbol{u}_2^*$ correspond to the bottom slope parameter and the eulerian velocity in layer two respectively.

There are many different dynamical limits that can be examined in the above set of equations. The scaling that we introduce will implicitly assume that both layers are, to leading order, in geostrophic balance in both the along-shore and offshore direction. The eulerian velocity field in the lower layer is scaled, following Flierl (1984), assuming that changes in relative vorticity are in balance with the production of vorticity from vortex stretching. The topographic slope will be scaled assuming it makes an equal contribution to the vorticity balance as these primary effects. The scalings we adopt for the upper layer will imply a somewhat weaker nonlinearity in the momentum equations compared with those presented in Flierl (1984).

The non-dimensional (unasterisked) variables are given by

$$\left.\begin{aligned} (x^*, y^*) = L(x, y), \quad t^* = t/(f_0\delta), \quad \eta^* = \delta^{\frac{1}{2}}(f_0^2 L^2/g)\,\eta, \\ \boldsymbol{u}_1^* = \delta^{\frac{1}{2}} f_0 L \boldsymbol{u}_1, \quad p^* = \rho_2 \delta f_0^2 L^2 p, \quad \boldsymbol{u}_2^* = \delta f_0 L \boldsymbol{u}_2, \end{aligned}\right\} \tag{2.4}$$

where the horizontal lengthscale is given by $L \equiv (g'H\delta^{\frac{1}{2}})^{\frac{1}{2}}/f_0$ or equivalently $L \equiv (g'h_*/\delta^{\frac{1}{2}})^{\frac{1}{2}}/f_0$ where h_* is a representative thickness scale for h^* and the reduced gravity is given by $g' = (\rho_2 - \rho_1)g/\rho_2$. The parameter δ will play the role of a vortex stretching parameter and is given by

$$\delta \equiv h_*/H. \tag{2.5a}$$

We also introduce a *scaled* slope parameter s which will measure the magnitude of the background vorticity gradient associated with the sloping bottom compared with baroclinic stretching and is given by

$$s \equiv s^* L/h_*. \tag{2.5b}$$

Substitution of the scaling (2.4) and (2.5) into (2.1), (2.2) and (2.3) yields, after a little algebra,

$$\delta \boldsymbol{u}_{1_t} + \delta^{\frac{1}{2}} \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 + \hat{e}_3 \times \boldsymbol{u}_1 + \nabla \eta = \boldsymbol{0}, \tag{2.6a}$$

$$\delta^{\frac{1}{2}} h_t + \nabla \cdot (\boldsymbol{u}_1 h) = 0, \tag{2.6b}$$

$$\delta(\partial_t + \boldsymbol{u}_2 \cdot \nabla)\, \boldsymbol{u}_2 + \hat{e}_3 \times \boldsymbol{u}_2 + \nabla p = \boldsymbol{0}, \tag{2.7a}$$

$$\nabla \cdot \boldsymbol{u}_2 = \delta h_t + \delta \nabla \cdot [\boldsymbol{u}_2 (h - sy)], \tag{2.7b}$$

$$\eta = h + \delta^{\frac{1}{2}} p, \tag{2.8}$$

where we have neglected terms of $O(g'/g)$ in the hydrostatic relation (2.8) to be consistent with the rigid-lid approximation. There are several remarks that should be made about these scaled equations.

The key parameter in the asymptotic analysis that follows is δ. As mentioned previously, this parameter will measure qualitatively the magnitude of the vortex stretching in the lower layer induced by a perturbed upper layer. The parameter s will be formally assumed as $O(1)$. This means that the generation of relative vorticity in the lower layer by baroclinic stretching will be balanced by the stabilizing influence of the topographic vorticity gradient. We should compare the equations we have obtained with analogous models obtained previously by others. The scaling we adopted was motivated in large part by Flierl (1984) and Swaters (1991). Our non-dimensionalization scheme can be interpreted as a slightly less nonlinear scaling than that used in Flierl (1984). It can be shown that our $\boldsymbol{u}_1(x, y, t)$ field is $O(\delta^{\frac{1}{2}})$ compared with Flierl's. On the other hand, our scalings correspond to a more nonlinear model than that presented in Swaters (1991) for cold-core fronts. However, it is important to add that while the model presented here is more nonlinear than the Swaters (1991) theory, the earlier work also had the upper and lower layers interacting at $O(1)$ and not $O(\delta^{\frac{1}{2}})$ as in this model. As it turns out, the scalings introduced here are forced on us if we want to construct a model that incorporates mean flow kinetic energy release, baroclinic vortex tube stretching and a topographic vorticity gradient within a context of $O(1)$ isopycnal deflections. In the barotropic limit corresponding to an infinitely deep motionless lower layer, the model reduces to only (2.6a, b) and (2.8) with $p \equiv 0$. If we interpret δ as a non-dimensional Rossby number, this reduced model is very similar to the governing equations presented by Cushman-Roisin (1986). One final aspect of the non-dimensionalizations that is worth pointing out is that our scalings correspond to intermediate lengthscale dynamics as described by Charney & Flierl (1981). The lengthscale adopted in (2.4) is $O(\delta^{-\frac{1}{4}})$ times the internal deformation radius associated with the upper layer. Consequently, under a weakly baroclinic limit, i.e. $0 < \delta \ll 1$,

$$(g' h_*)^{\frac{1}{2}}/f_0 \ll L \ll (g' H)^{\frac{1}{2}}/f_0.$$

The weakly baroclinic limit is physically relevant. Typically values on a continental shelf might be $s^* \approx 1.2$ m km^{-1}, $h_* \approx 30$–40 m, $H \approx 200$–300 m and $(g'H)^{\frac{1}{2}}/f_0 \approx 15$ km suggesting $L \approx 10$ km, $\delta \approx 2 \times 10^{-2}$ and $s \approx 0.4$; see also the discussion in Swaters & Flierl (1991) and Swaters (1991).

To examine the small δ limit in (2.6), (2.7) and (2.8), we proceed with a straightforward asymptotic expansion of the form

$$(\eta, h, p, \boldsymbol{u}_1, \boldsymbol{u}_2) \sim (\eta, h, p, \boldsymbol{u}_1, \boldsymbol{u}_2)^{(0)} + \delta^{\frac{1}{2}} (\eta, h, p, \boldsymbol{u}_1, \boldsymbol{u}_2)^{(1)} + O(\delta). \tag{2.9}$$

The details of the expansion for the slope water equations are straightforward when we observe that the location of the parameter δ in (2.7 *a*, *b*) occurs in such a way that the dynamics is quasigeostrophic to $O(\delta)$. Hence the evolution of the leading order flow is determined by

$$(\Delta p^{(0)}+h^{(0)})_t-sp_x^{(0)}+J(p^{(0)},\Delta p^{(0)}+h^{(0)})=0, \tag{2.10a}$$

where $\Delta\equiv\partial^2/\partial x^2+\partial^2/\partial y^2$ and $J(A,B)=A_xB_y-A_yB_x$. The leading order velocity field is geostrophically determined and thus given by

$$\boldsymbol{u}_2^{(0)}=\hat{e}_3\times\nabla p^{(0)}. \tag{2.10b}$$

The details of the expansion for the frontal flow are slightly more subtle. The $O(1)$ and $O(\delta^{\frac{1}{2}})$ problems associated with (2.6) and (2.8) are given by, respectively,

$$\boldsymbol{u}_1^{(0)}=\hat{e}_3\times\nabla h^{(0)},\quad \nabla\cdot[\boldsymbol{u}_1^{(0)}h^{(0)}]=0,\quad \eta^{(0)}=h^{(0)}, \tag{2.11a–c}$$

$$\hat{e}_3\times\boldsymbol{u}_1^{(1)}+\nabla\eta^{(1)}=-(\boldsymbol{u}_1^{(0)}\cdot\nabla)\,\boldsymbol{u}_1^{(0)}, \tag{2.12a}$$

$$h_t^{(0)}+\nabla\cdot[\boldsymbol{u}_1^{(0)}h^{(1)}+\boldsymbol{u}_1^{(1)}h^{(0)}]=0, \tag{2.12b}$$

$$\eta^{(1)}=h^{(1)}+p^{(0)}. \tag{2.12c}$$

The equations (2.11) are not enough to specify the $O(1)$ solution since substitution of (2.11 *a*) into (2.11 *b*) will automatically be satisfied for all sufficiently smooth $h^{(0)}$. Note, however, that no restriction is made on the magnitude of $h^{(0)}$. As well, to leading order, it is important to note that $\eta^{(0)}$ is decoupled from $p^{(0)}$.

Substitution of (2.11 *a*) into (2.12 *a*) leads to, after a little algebra,

$$\boldsymbol{u}_1^{(1)}=\hat{e}_3\times\nabla\eta^{(1)}+J(\nabla h^{(0)},h^{(0)}). \tag{2.13}$$

The first term on the right-hand side of (2.13) is simply the perturbation geostrophic flow and the second term is the contribution to $\boldsymbol{u}^{(1)}$ from the nonlinear momentum terms in (2.12 *a*). Further substitution of (2.11 *a*) and (2.13) into (2.12 *b*), after eliminating $\eta^{(1)}$ in favour of $h^{(1)}$ and $p^{(0)}$ via (2.12 *c*), yields

$$h_t^{(0)}+J(p^{(0)}+h^{(0)}\Delta h^{(0)}+\tfrac{1}{2}\nabla h^{(0)}\cdot\nabla h^{(0)},h^{(0)})=0. \tag{2.14}$$

The pair of equations (2.10 *a*) and (2.14) determine the coupled evolution of $h^{(0)}$ and $p^{(0)}$. The leading order velocity field will be given by (2.10 *b*) and (2.11 *a*), for the slope water and frontal layer, respectively, and the leading dynamic pressure in the frontal layer will be given by (2.11 *c*). The barotropic limit of (2.14), corresponding to letting $p^{(0)}\to 0$, is identical to the Cushman-Roisin (1986) frontal model restricted to a *f*-plane. (I am aware that a model somewhat similar to (2.10 *a*) and (2.14) has been independently obtained by Cushman-Roisin *et al.* (1992) in a somewhat different context appropriate for a midlatitude β-plane and has been used to study aspects of geostrophic turbulence (Tang & Cushman-Roisin 1992).)

(*b*) *Boundary conditions*

In addition to appropriate boundary conditions on the coast and for the distant offshore, dynamic and kinematic conditions are required for outcroppings (i.e. places where $h(x,y,t)\equiv 0$). The physical problem we wish to examine here is depicted in figure 1, wherein the fluid that is immediately adjacent to the coast is slope water (in contrast to the problem examined by Killworth and Stern). To leading order the no normal flow condition is simply

$$p_x^{(0)}=0 \quad \text{on} \quad y=-B, \tag{2.15a}$$

and for offshore we require a bounded velocity field

$$\left.\begin{array}{l}|\nabla p^{(0)}| < \infty,\\ |\nabla h^{(0)}| < \infty,\end{array}\right\} \quad \text{as} \quad y \to \infty, \tag{2.15b, c}$$

(although for non-leaky modes (2.15*b*) can be replaced with $p^{(0)} \to 0$ and $h^{(0)} \to 0$ as $y \to \infty$).

The boundary conditions for the outcroppings are obtained as follows. Suppose the projection on $z^* = 0$ of an outcropping of the front is *dimensionally* given by the curve $y^* = \phi^*(x^*, t^*)$. If we introduce the scalings (2.4) and define $\phi(x,t) \equiv \phi^*(x^*, t^*)/L$, the *non-dimensional* kinematic condition can be written in the form

$$\delta^{\frac{1}{2}} \phi_t + u_1 \phi_x = v_1 \quad \text{on} \quad y = \phi(x,t), \tag{2.16a}$$

and the frontal height satisfies

$$h = 0 \quad \text{on} \quad y = \phi(x,t). \tag{2.16b}$$

Inserting the expansion (2.9) together with

$$\phi \sim \phi^{(0)} + \delta^{\frac{1}{2}} \phi^{(1)} + \dots, \tag{2.17}$$

into (2.16) yields the $O(1)$ conditions

$$h_x^{(0)} + h_y^{(0)} \phi_x^{(0)} = 0, \quad h^{(0)} = 0, \tag{2.18a, b}$$

on $y = \phi^{(0)}(x,t)$.

The $O(\delta^{\frac{1}{2}})$ boundary conditions take the form

$$\phi_t^{(0)} + u_1^{(1)} \phi_x^{(0)} + u_1^{(0)} \phi_x^{(1)} + \phi_x^{(0)} \phi^{(1)} u_{1_y}^{(0)} = v_1^{(1)} + \phi^{(1)} v_y^{(0)}, \tag{2.19a}$$

$$h^{(1)} + h_y^{(0)} \phi^{(1)} = 0, \tag{2.19b}$$

on $y = \phi^{(0)}(x,t)$. Substitution of (2.13) into (2.19*a*) and eliminating $\eta^{(1)}(x, \phi, t)$ using (2.12c) and $\phi^{(1)}(x,t)$ using (2.19*a*) leads to

$$h_y^{(0)} \phi_t^{(0)} = J(p^{(0)}, h^{(0)}) + \nabla h^{(0)} \cdot J(\nabla h^{(0)}, h^{(0)}), \tag{2.20}$$

on $y = \phi^{(0)}(x,t)$. Assuming that we can take the limit $y \to \phi(x,t)$ smoothly in (2.14), allows (2.20) to simplify to

$$h_y^{(0)} \phi_t^{(0)} + h_t^{(0)} = 0, \quad \text{on} \quad y = \phi^{(0)}(x,t). \tag{2.21}$$

In addition to these conditions, we also require that the slope-water pressure and normal mass flux be continuous across $y = \phi^{(0)}(x,t)$.

The model equations (2.10*a*) and (2.14) together with the boundary conditions (2.15*a*, *b*), (2.18) and (2.21) possess an exact nonlinear along shelf-solution of the form

$$h^{(0)} = h_0(y), \quad p^{(0)} = p_0(y) \equiv -\int_0^y U_0(\xi)\, \mathrm{d}\xi, \tag{2.22a, b}$$

$$\phi^{(0)} = a, \tag{2.23}$$

where a is constant in (2.23), and where it is assumed that $h_0(a) = 0$. In the case of multiple outcroppings, such as a coupled front, (2.23) would be replaced with a set of similar relations, one for each outcrop location. Because we are free to choose the origin of the coordinate system where we wish, we can choose $a \equiv 0$ in (2.23), so that $h_0 \equiv 0$ in the region $-B \leqslant y \leqslant 0$ (see figure 1).

(*c*) *Hamiltonian formulation, Casimirs and general steady solutions*

Over the last several years infinite-dimensional hamiltonian theory has been applied to fluid mechanics in order to make very general and deep statements about the underlying structure and stability of fluid flows (see, for example, Arnol'd 1969; Olver 1982; Benjamin 1984; Holm *et al.* 1985, among many others); a readable review appropriate to GFD can be found in Shepherd (1990). In this section we show how the model equations just derived can be put in non-canonical hamiltonian form. We then compute the appropriate Poisson bracket for our model and obtain the family of Casimir functionals for the dynamics. Finally, we show how to construct a constrained hamiltonian functional so that general steady solutions of the model will satisfy the first-order necessary conditions for extremizing the constrained hamiltonian. These results will be needed in §4 where we generalize the linear stability theorems obtained in §3 for along-shelf steady shear-flows of the form (2.22) and (2.23) to more general steady solutions, and to a nonlinear stability analysis.

The model together with boundary and matching conditions derived in §2 can be written in the slightly generalized form (dropping the (0)-superscript)

$$(\Delta p+h)_t - sp_x + J(p, \Delta p+h) = 0, \tag{2.24}$$

$$h_t + J(p+h\Delta h+\tfrac{1}{2}\nabla h\cdot\nabla h, h) = 0, \tag{2.25}$$

where the two-dimensional spatial domain associated with $p(x,y,t)$ will be abstractly denoted $R \subset \mathbb{R}^2$ and the two-dimensional frontal region associated with $h(x,y,t)>0$ will be denoted $F \subset R$. The boundary of R will be denoted ∂R. The outcropping will occur on the boundary of F which will be denoted ∂F or equivalently by $y=\phi(x,t)$ as in §2.

The boundary conditions on $p(x,y,t)$ therefore take the form

$$\partial p/\partial s = 0 \quad \text{on} \quad \partial R, \tag{2.26}$$

where s is arclength along ∂R, and the conservation of net circulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial R} \nabla p\cdot \boldsymbol{n}\,\mathrm{d}s = 0, \tag{2.27}$$

where n is the unit outward normal on ∂R in the case where R is bounded. In the case where R is the half-plane as discussed in §2, ∂R is not a closed curve and hence (2.27) plays no role and (2.26) reduces to (2.15*a*) for the coast. On any outcropping

$$h(x,\phi(x,t),t) = 0, \tag{2.28}$$

It is important to note that the boundary conditions (2.18*a*) and (2.21) play no role in the dynamics since under the assumption that $h(x,y,t)$ is formally differentiable as $(x,y)\to\partial F$ it follows from (2.28) that (2.18*a*) and (2.21) automatically hold. This will not lead to an ill-posed mathematical problem since the solutions will be uniquely determined by demanding that ∇p be continuous across $y=\phi(x,t)$ (in the context of the linear problem for cold-core fronts see how Swaters (1991) resolved this issue). This fact has the very important implication that we can avoid all of the major technical difficulties associated with hamiltonian formulations of free-boundary problems (Lewis *et al.* 1986).

A system of partial differential equations is hamiltonian if it can be written in the form (Olver 1982)

$$\boldsymbol{q}_t = \mathscr{D}\,\delta H/\delta \boldsymbol{q}, \tag{2.29}$$

where $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)^{\mathrm{T}}$ is a column vector of n dependent variables, $H(\boldsymbol{q})$ is the hamiltonian functional (conserved but not necessarily positive definite or even the area-integrated energy; although it often is) and $\delta H/\delta \boldsymbol{q}$ are the Euler derivatives of H with respect to $\boldsymbol{q}$, $\mathscr{D}$ is a skew-symmetric matrix of differential operators satisfying

$$(*, \mathscr{D} \cdot) = -(\mathscr{D} *, \cdot), \tag{2.30}$$

where $(*, \cdot)$ is the appropriate inner product for the phase space in question, and $\mathscr{D}$ must satisfy the Jacobi identity as well.

Theorem 2.1. *The equations* (2.24) *and* (2.25) *can be put into the following hamiltonian form. Define* $\boldsymbol{q} = (q_1, q_2)^{\mathrm{T}}$ *where*

$$q_1 = \Delta p + h, \quad q_2 = h, \tag{2.31\,a, b}$$

with the hamiltonian given by

$$H = \frac{1}{2}\iint_R \nabla p \cdot \nabla p \,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\iint_F h \nabla h \cdot \nabla h \,\mathrm{d}x\,\mathrm{d}y - \lambda \oint_{\partial R} \nabla p \cdot \boldsymbol{n} \,\mathrm{d}s, \tag{2.32}$$

where λ *is the value of* p *on* ∂R. *In the case where* R *is simply-connected we may set* $\lambda \equiv 0$. *In the case where* R *is not simply-connected the last term is to be replaced with a sum over all the closed boundary curves. In the case where* R *is unbounded the last integral must be interpreted via an appropriate limit argument. The matrix* $\mathscr{D} = [\mathscr{D}_{ij}]$ *is given by*

$$\mathscr{D}_{ij} = -\delta_{i1}\,\delta_{j1}\,J(q_1 - sy, *) + \delta_{2i}\,\delta_{2j}\,J(q_2, *), \tag{2.33}$$

where δ_{ij} *is the Kronecker delta function for* $1 \leqslant i, j \leqslant 2$.

To show that H is conserved we formally compute $\partial H/\partial t$. Kelvin's circulation theorem (2.27) eliminates the last term in (2.32). The remaining terms can be shown to vanish by direct substitution of (2.24) and (2.25) into $\partial H/\partial t$, and integrating by parts exploiting the boundary conditions (2.28) and the smoothness of $p(x, y, t)$.

The variational derivatives are obtained from the first variation of H given by

$$\delta H = \iint_R \nabla p \cdot \nabla \delta p \,\mathrm{d}x\,\mathrm{d}y - \iint_F \delta h \tfrac{1}{2} \nabla h \cdot \nabla h + h \nabla h \cdot \nabla\,\delta h \,\mathrm{d}x\,\mathrm{d}y - \lambda \oint_{\partial R} \boldsymbol{n} \cdot \nabla\,\delta p \,\mathrm{d}s, \tag{2.34}$$

where the contribution associated with δF vanishes since $h \equiv 0$ on δF. Integrating the first two integrals by parts and exploiting the boundary conditions leads to

$$\delta H = \iint_R (-p)\,\Delta \delta p \,\mathrm{d}x\,\mathrm{d}y + \iint_F (h \Delta h + \tfrac{1}{2} \nabla h \cdot \nabla h)\,\delta h \,\mathrm{d}x\,\mathrm{d}y,$$

or equivalently,

$$\delta H = \iint_R (-p)\,(\Delta \delta p + \delta h) \,\mathrm{d}x\,\mathrm{d}y + \iint_F (h \Delta h + \tfrac{1}{2} \nabla h \cdot \nabla h + p)\,\delta h \,\mathrm{d}x\,\mathrm{d}y, \tag{2.35}$$

where it is implicitly understood that δh is only non-zero in the domain F in the first integral. Since $\delta q_1 \equiv \Delta \delta p + \delta h$ and $\delta q_2 \equiv \delta h$, it follows from (2.35) that

$$\delta H/\delta \boldsymbol{q} \equiv (-p, h \Delta h + \tfrac{1}{2} \nabla h \cdot \nabla h + p)^{\mathrm{T}}. \tag{2.36}$$

Substitution of (2.36) into (2.29) yields

$$\begin{aligned}(\Delta p + h)_t &= -J(\Delta p + h - sy, -p), \\ h_t &= J(h, h \nabla h + \tfrac{1}{2} \nabla h \cdot \nabla h + p),\end{aligned}$$

which can be rearranged into (2.24) and (2.25) thus verifying that the dynamics is reproduced by (2.29) and (2.32).

We can equivalently cast the dynamics using a Poisson bracket (Benjamin 1984) defined formally through

$$[\mathscr{F},\mathscr{H}] \equiv (\delta\mathscr{F}/\delta\boldsymbol{q}, \mathscr{D}\,\delta\mathscr{H}/\delta\boldsymbol{q}), \tag{2.37a}$$

where $\mathscr{F}$ and $\mathscr{H}$ are arbitrary functionals with densities depending on $\boldsymbol{q}$ and where the inner product is simply given by

$$(\boldsymbol{f},\boldsymbol{g}) \equiv \iint (f_1 g_1 + f_2 g_2)\,\mathrm{d}x\,\mathrm{d}y.$$

In terms of the Poisson bracket (2.37*a*), the evolution of the functional F is determined by

$$\mathrm{d}\mathscr{F}/\mathrm{d}t = [\mathscr{F}, H]. \tag{2.37b}$$

Thus the dynamics can be re-written as

$$\boldsymbol{q}_t = [\boldsymbol{q}, H], \tag{2.37c}$$

provided we interpret (2.37*c*) in component form and

$$\boldsymbol{q} \equiv \iint \delta(x-x')\delta(y-y')\,\boldsymbol{q}(x',y',t)\,\mathrm{d}x'\,\mathrm{d}y'. \tag{2.37d}$$

The Casimir functionals are defined to be the family of invariants which span the kernel of the Poisson bracket. In canonical hamiltonian dynamics, in which $\mathscr{D}$ is non-singular, the Casimirs are trivial. However, in our situation $\mathscr{D}$ is singular and thus the Casimirs are non-trivial. If we define $C = C(\boldsymbol{q})$ to be a Casimir functional, then

$$0 = [\mathscr{F}, C] \equiv (\delta\mathscr{F}/\delta\boldsymbol{q}, \mathscr{D}\,\delta C/\delta\boldsymbol{q}), \tag{2.38}$$

where $\mathscr{F}(\boldsymbol{q})$ is an arbitrary functional; that is, they are the solutions of

$$\mathscr{D}\,\delta C/\delta\boldsymbol{q} = \boldsymbol{0},$$

or equivalently,

$$J(q_1 - sy, \delta C/\delta q_1) = 0, \quad J(q_2, \delta C/\delta q_2) = 0. \tag{2.39a, b}$$

The general solution to (2.39) is given by

$$C = \iint_R \Phi_1(\Delta p + h - sy)\,\mathrm{d}x\,\mathrm{d}y + \iint_F \Phi_2(h)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.40}$$

where Φ_1 and Φ_2 are arbitrary functions of the potential vorticity and frontal thickness, respectively, and where it is implicitly understood that h is non-zero only in domain F in the first integral.

We can use the Casimirs to show that arbitrary steady solutions of (2.24) and (2.25) will satisfy the first-order necessary conditions for a conditional extremal of the hamiltonian H. It follows from (2.24) and (2.25) that *general* steady solutions given by $p \equiv p_0(x,y)$ and $h \equiv h_0(x,y)$ satisfy

$$J(p_0, \Delta p_0 + h_0 - sy) = 0,$$
$$J(p_0 + h_0\,\Delta h_0 + \tfrac{1}{2}\nabla h_0 \cdot \nabla h_0, h_0) = 0,$$

which integrate to

$$p_0 = F_1(\Delta p_0 + h_0 - sy), \tag{2.41}$$

$$p_0 + h_0 \Delta h_0 + \tfrac{1}{2}\nabla h_0 \cdot \nabla h_0 = F_2(h_0), \tag{2.42}$$

where F_1 and F_2 are suitably smooth arbitrary functions with respect to their respective arguments. If we consider the constrained hamiltonian

$$\tilde{H}(p,h) \equiv H(p,h) + C(p,h), \tag{2.43}$$

where H is given by (2.32) and C by (2.40), it follows that the first variation is given by

$$\delta\tilde{H} = -\iint_R p\Delta\delta p \, dx\, dy + \iint_F (h\Delta h + \tfrac{1}{2}\nabla h \cdot \nabla h)\, \delta h \, dx\, dy + \iint_R (\Delta\delta p + \delta h)\, \Phi_1' \, dx\, dy + \iint_F \delta h \Phi_2' \, dx\, dy,$$

where $\Phi_1' \equiv d\Phi_1'/d(\Delta p + h - sy)$ and $\Phi_2' \equiv d\Phi_2/dh$. This expression can be rewritten as

$$\delta\tilde{H}(p,h) = \iint_R (\Phi_1' - p)(\Delta\delta p + \delta h)\, dx\, dy + \iint_F (\Phi_2' + p + h\Delta h + \tfrac{1}{2}\nabla h \cdot \nabla h)\, \delta h \, dx\, dy. \tag{2.44}$$

Hence we see that

$$\delta\tilde{H}(p_0, h_0) = 0, \tag{2.45}$$

provided we choose the Casimir density as

$$\Phi_1 \equiv \int^{\Delta p + h - sy} F_1(\xi)\, d\xi, \quad \Phi_2 \equiv -\int^{h} F_2(\xi)\, d\xi, \tag{2.46a, b}$$

where F_1 and F_2 are given by (2.41) and (2.42) respectively. To summarize, arbitrary steady solutions to (2.24) and (2.25) defined through the relations (2.41) and (2.42) satisfy the first-order necessary condition (2.45) for extremizing the constrained hamiltonian (2.43) provided the Casimir (2.40) is chosen to satisfy (2.46). We can use the constrained hamiltonian $\tilde{H}$ to derive conditions for the linear stability of arbitrary steady solutions and a generalization of $\tilde{H}$ to examine the nonlinear stability of these steady solutions. This is done in §4. Before doing this however, it is very useful to study the general linear stability characteristics of steady density-driven currents and fronts which have an along-shore configuration. In practice, it is to be expected that this class of steady solutions will be among the most useful from the point of view of applying our theory to density-driven currents and fronts of coastal oceanographic interest.

3. Linear stability problem and general stability characteristics

(a) *Linear stability equations and boundary conditions*

In this section we want to examine in some detail the linear stability properties associated with steady along-shore solutions given by (2.22) and (2.23). The results of a linear stability analysis for these along-shore solutions can be very useful with respect to motivating a general linear stability analysis and leading the way to a finite-amplitude stability theory which requires the results of the hamiltonian structure developed in the last section.

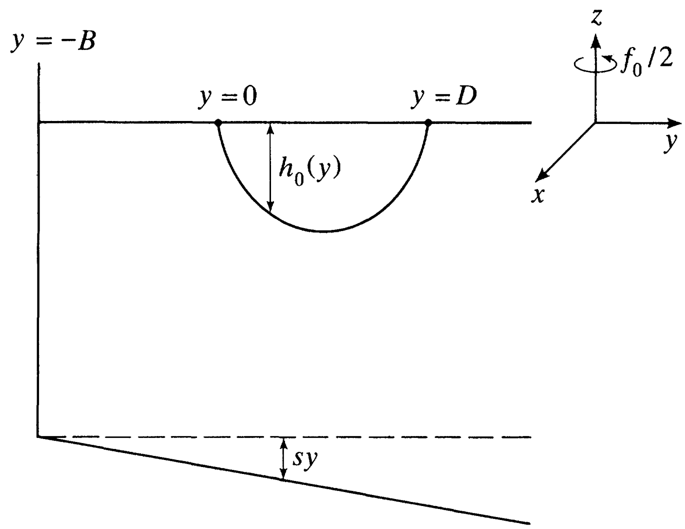

Figure 2. Geometry of the linear instability problem examined in §3.

To examine the linear stability problem for the along-shore solutions we assume that

$$h = h_0(y) + h'(x,y,t), \quad p = p_0(y) + p'(x,y,t), \quad \phi = a + \phi'(x,t), \qquad (3.1\,a\text{–}c)$$

where $a = 0$ or $a = D$ (if $D < \infty$; see figure 2 for the geometry assumed in this section) and substitute into the model equations and linearize about the $h_0(y)$ and $p_0(y)$.

In the frontal region $0 < y < D$, the stability problem takes the form (after dropping the prime notation for the perturbation fields)

$$h_{0y}\,h_0\,\Delta h_x + (h_{0y})^2 h_{xy} + [U_0 - (h_0\,h_{0yy})_y]\,h_x + h_t + h_{0y}\,p_x = 0, \qquad (3.2\,a)$$

$$(\partial_t + U_0\,\partial_x)\,(\Delta p + h) + (h_{0y} - U_{0yy} - s)\,p_x = 0. \qquad (3.2\,b)$$

In the non-frontal regions $-B < y < 0$ and $D < y < \infty$, the stability problem takes the form

$$(\partial_t + U_0\,\partial_x)\,\Delta p - (U_{0yy} + s)\,p_x = 0. \qquad (3.3)$$

Equation (3.3) is simply a Rayleigh stability equation including the effects of a linearly sloping bottom.

The linearized and Taylor-expanded boundary conditions are given as follows. From (2.18*a*, *b*) we have

$$h_x + h_{0y}\,\phi_x = 0, \quad h + h_{0y}\,\phi = 0, \qquad (3.4\,a, b)$$

on $y = 0$ respectively. From (2.21) we have

$$h_t + h_{0y}\,\phi_t = 0, \qquad (3.4\,c)$$

on $y = 0$. From (2.15*a*, *b*) we have

$$p_x = 0 \quad \text{on} \quad y = -B, \qquad (3.4\,d)$$

$$\left.\begin{aligned} |\nabla p| &< \infty, \\ |\nabla h| &< \infty, \end{aligned}\right\} \quad \text{as} \quad y \to \infty, \qquad (3.4\,e,f)$$

(where (3.4*f*) is needed only if $D = \infty$) respectively. We can see immediately that (3.4*a*–*c*) are degenerate in the sense that if (3.4*b*) holds on $y = 0$, it immediately follows that for sufficiently smooth $h(x,t)$ and $\phi(x,t)$, (3.4*a*) and (3.4*c*) also hold since h_{0y} is only a function of the cross-shelf coordinate y. This means that the boundary conditions (3.4a–c) are not enough to uniquely specify a solution to (3.2) and (3.3) together with the remaining boundary conditions (3.4d–f). This apparent difficulty is obviated by the condition that the pressure and normal mass flux in the slope water must be continuous across $y = 0$.

(b) *Perturbation energetics*

We can obtain some general qualitative information on the nature of the instabilities by examining an averaged energy equation associated with (3.2a). If (3.2a) is multiplied through by $h(x, y, t)$ and the result integrated over $0 < y < D$ and $0 < x < \lambda$ where λ is the along-shelf wavelength of the perturbation and where the steady current is assumed to occupy the region $y \in (0, D)$ where $0 < D \leqslant \infty$, it follows after repeated integration by parts, that

$$\frac{\partial}{\partial t}\int_0^D \tfrac{1}{2}\langle h^2\rangle \,\mathrm{d}y = -\int_0^D \tau h_{0yy}\,\mathrm{d}y - \int_0^D h_{0y}\langle hp_x\rangle\,\mathrm{d}y, \tag{3.5a}$$

where τ is the along-shore averaged perturbation Reynolds stress given by $\tau \equiv h_0\langle h_y h_x\rangle$ where $\langle(*)\rangle$ is defined as

$$\langle(*)\rangle = \lambda^{-1}\int_0^\lambda (*)\,\mathrm{d}x. \tag{3.6}$$

In the 'pure' barotropic problem (i.e. $p \equiv 0$ in (3.5)), instability can only occur if τ is on average negatively correlated with the frontal vorticity h_{0yy}. That is, of course, just the classical shear flow instability result. However, as it turns out, there is no linear instability in the pure barotropic problem. This can be easily seen as follows. The *barotropic* limit of the linear instability equations (3.2) corresponds to setting $U_0 \equiv p \equiv 0$ in (3.2a) and ignoring (3.2b) altogether, that is,

$$h_t - (h_0 h_{0yy})_y h_x + (h_{0y})^2 h_{xy} + h_{0y} h_0 \Delta h_x = 0.$$

It can be shown that this equation possesses the positive-definite quadratic invariant

$$L = \int_0^D h_0(h_{0y})^2\langle\nabla(h/h_{0y})\cdot\nabla(h/h_{0y})\rangle\,\mathrm{d}y.$$

(We show in §4 that $-L$ is the barotropic limit of the second variation of the constrained hamiltonian (2.43) evaluated for the steady solution (2.22a); see (4.19).) Since $\partial L/\partial t \equiv 0$, it follows that $L(t) \equiv L(0) > 0$ and consequently if $L(0)$ is bounded so is $L(t)$ and $h(x, y, t)$ for all $t \geqslant 0$ thus establishing linear stability in the sense of Liapunov for the barotropic problem with respect to the norm $\|h\|^2 \equiv L$. This is an important result because it implies that baroclinic coupling in this model is necessary for any instability to occur.

Returning now back to (3.5a), if baroclinic processes dominate, then it follows that instability can only occur if $\langle hp_x\rangle$ is *negatively* correlated, on average, with h_{0y}. It is possible to give a heuristic physical interpretation to the above baroclinic instability condition as follows. Because the flow in the slope-water is geostrophically, balanced, p_x is simply the cross-shelf perturbation velocity. We can interpret a positive h as a warm anomaly in the underlying slope water. Thus, qualitatively, baroclinic instability can only occur if, on average, the transport of warm anomalies in the slope water is in the onshore direction if $h_{0y} > 0$ and in the offshore direction if $h_{0y} < 0$. These conditions are exactly analogous to those presented in Swaters (1991).

(*c*) *The along-front normal mode equations*

In the remainder of this section we focus attention on normal mode instabilities of the form

$$h = \tilde{h}(y)\exp[\mathrm{i}k(x-ct)]+\text{c.c.}, \quad (3.7a)$$

$$p = \tilde{p}(y)\exp[\mathrm{i}k(x-ct)]+\text{c.c.}, \quad (3.7b)$$

$$\phi = \tilde{\phi}\exp[\mathrm{i}k(x-ct)]+\text{c.c.}, \quad (3.7c)$$

where k and c are the along-front wavenumber and complex phase speed, respectively, and c.c. denotes complex conjugate. Substitution of (3.7) into (3.2), (3.3) and (3.4) yields the problem (after dropping the tilde)

$$h_{0y}h_0h_{yy}+(h_{0y})^2h_y-[k^2h_0h_{0y}+(h_0h_{0yy})_y+c-U_0]h+h_{0y}p = 0, \quad (3.8a)$$

$$p_{yy}+[-k^2+(h_{0y}-U_{0yy}-s)(U_0-c)^{-1}]p+h = 0, \quad (3.8b)$$

for $0 < y < D$ and the problem

$$p_{yy}-[k^2+(U_{0yy}+s)(U_0-c)^{-1}]p = 0, \quad (3.8c)$$

in $-B < y < 0$ and $D < y < \infty$. The boundary conditions reduce to

$$h+h_{0y}\phi = 0, \quad \text{on} \quad y = 0, D, \quad p(-B) = 0, \quad (3.9a, b)$$

$$\left.\begin{aligned}|p|, |p_y| < \infty,\\ |h|, |h_y| < \infty,\end{aligned}\right\} \quad \text{as} \quad y \rightarrow \infty, \quad (3.9c, d)$$

where (3.9d) is needed if $D \equiv \infty$. The conditions that the pressure and normal mass flux in the slope water be continuous across $y = 0$ are given by

$$[(c-U_0)p_y+U_{0y}p] = 0, \quad [p(U_0-c)^{-1}] = 0, \quad (3.9e, f)$$

on $y = 0$, where $[(*)](y = 0) \equiv (*)(0^+)-(*)(0^-)$. Of course, (3.9) must also hold at any other location where it may happen that $U_0(y)$ or $U_{0y}(y)$ are discontinuous. In addition to these matching conditions, certain regularity conditions will have to be imposed on $h(y)$ at an outcropping since (3.8a) is singular there. These conditions are derived in §3e).

(*d*) *General results for the normal modes*

In contrast to the theory presented in Swaters (1991), we have not been able to obtain many rigorous general results for the normal-mode equations given here. However, as it turns out, it is possible to explicitly obtain two relatively simple stability conditions for the normal modes.

The most convenient way to obtain these results is to work with a new dependent variable in place of $h(y)$ that casts (3.8a, b) into self-adjoint form. To that end define $F(y)$ through

$$h(y) = h_{0y}F(y). \quad (3.10)$$

This transformation will certainly be nonsingular where $h_{0y} \neq 0$. However, we will need h and h_y to be finite at any location where $h_{0y} = 0$ because continuity requirements on the mass transports and pressure in the frontal layer. Thus the regularity of $F(y)$ is somewhat constrained.

Substitution of (3.10) into (3.8a, b) leads to

$$[h_0(h_{0y})^2F_y]_y-(k^2h_0h_{0y}+c-U_0)h_{0y}F+h_{0y}p = 0, \quad (3.11a)$$

$$p_{yy}-[k^2+(s+U_{0yy}-h_{0y})(U_0-c)^{-1}]p+h_{0y}F = 0. \quad (3.11b)$$

Multiplying the complex-conjugate of (3.11a) with F and integrating over the frontal region yields the balance

$$\int_0^D h_0(h_{0y})^2 [|F_y|^2 + k^2 |F|^2] + (c^* - U_0) h_{0y} |F|^2 \, dy = \int_0^D h_{0y} p^* F \, dy, \qquad (3.12a)$$

where p^* is the complex-conjugate of $p(y)$. Similarly, multiplying (3.11b) and (3.8c) by p^*, integrating over $(-B, \infty)$ and adding together yields the balance

$$\int_{-B}^{\infty} |p_y|^2 + [k^2 + (U_0 - c^*)(s + U_{0yy}) |U_0 - c|^{-2}] |p|^2 \, dy$$
$$+ \int_0^D (c^* U_0) h_{0y} |U_0 - c|^2 |p|^2 \, dy = \int_0^D h_{0y} F p^* \, dy. \qquad (3.12b)$$

If we eliminate the right-hand side of (3.12b) using (3.12a), we obtain

$$\int_{-B}^{\infty} |p_y|^2 + [k^2 + (U_0 - c^*)(s + U_{0yy}) |U_0 - c|^{-2}] |p|^2 \, dy$$
$$+ \int_0^D (c^* - U_0) h_{0y} |U_0 - c|^{-2} |p|^2 \, dy = \int_0^D h_0 h_{0y} [|F_y|^2 + k^2 |F|^2] + (c^* - U_0) h_{0y} |F|^2 \, dy. \qquad (3.13)$$

The imaginary part of (3.13) is given by (after substituting $c \equiv c_{\mathrm{R}} + \mathrm{i}c_{\mathrm{I}}$)

$$c_{\mathrm{I}} \left\{ \int_0^D h_{0y} |F|^2 \, dy + \int_{-B}^{\infty} [s + U_{0yy} - \Theta h_{0y}] |U_0 - c|^{-2} |p|^2 \, dy \right\} = 0, \qquad (3.14)$$

where $\theta = 1$ for $y \in (0, D)$ and $\theta = 0$ for $y \in (-B, 0) \cup (D, \infty)$. Clearly, if $\{*\} \neq 0$ for all modal solutions, then $c_{\mathrm{I}} = 0$ so we have neutral stability. There are two easily identifiable stability theorems given as follows.

Theorem 3.1. *If $h_{0y} > 0$ for all $y \in (0, D)$ (i.e. a monotonic frontal depth profile) and $0 \leqslant h_{0y} < U_{0yy} + s$ for all $y \in (-B, \infty)$, then the front is neutrally stable.*

These conditions ensure that the quantity inside the curly brackets is positive definite for all non-trivial solutions, thus implying $c_{\mathrm{I}} \equiv 0$. In the pure baroclinic limit for which $U_0 \equiv 0$, these two conditions reduce to the single stability condition

$$0 < h_{0y} < s,$$

for all $y \in (0, D)$. Hence in the pure baroclinic limit, a necessary condition for instability is the existence of at least one y-value for which either $h_{0y} < 0$ or $h_{0y} > s$. The latter condition shows the stabilizing influence of the bottom slope for a monotonic frontal profile satisfying $h_{0y} > 0$ everywhere. Even if $s > h_{0y}$, instability has the possibility of occurring provided $h_{0y} < 0$ somewhere. Physically, this can occur for a coupled-front configuration where $h_0(y)$ contains two outcroppings. On the offshore side of such a current it follows $h_{0y} < 0$ (see Swaters 1991). This necessary condition for instability can be easily interpreted in terms of the potential vorticity of the front. The non-dimensional potential vorticity of the front is given by $PV = (\delta^{\frac{1}{2}} \nabla x \boldsymbol{u}_1 + 1)/h$. In the limit as $\delta \rightarrow 0$ we have $PV \sim 1/h^{(0)}$ so that for the mean flow $h_{0y} < 0 \Rightarrow (PV)_y \sim -h_{0y}/h_0^2 > 0$. Therefore a necessary condition for instability in the situation where $s > h_{0y}$ is that the leading order potential vorticity associated with the front must increase in the offshore direction for some values of y.

Theorem 3.2. *If $h_{0y} < 0$ for all $y \in (0, D)$ and $U_{0yy} + s < h_{0y} \leqslant 0$ for all $y \in (-B, \infty)$, then the front is neutrally stable.*

These conditions will imply that the quantity inside the curly brackets will be negative definite for all non-trivial solutions, thus implying $c_I \equiv 0$. It is physically realistic to have a frontal profile satisfying $h_{0y} < 0$. For example, consider a buoyancy-driven front along a coast which has maximum h_0 on the coast and for which h_0 monotonically decreases to zero offshore. Another important point to make about this stability condition is that it only makes sense if a background mean flow $U_0(y)$ is present with a sufficiently negative vorticity gradient since if ever $U_{0yy} + s > 0$ (such as in the pure baroclinic limit $U_0 \equiv 0$), it will be impossible to satisfy the second of the above two conditions.

It is important to understand that the above stability theorems do not imply that a zero in the cross-shelf potential vorticity gradients are necessary for instability. For example, (3.14) can be satisfied in principle if the leading order (as $\delta \to 0$) potential vorticity gradients in each layer given by

$$(PV_1)_y \approx -h_{0y}/h_0^2, \quad (PV_2)_y \approx -(U_{0yy} + sy - h_{0y}),$$

are of constant but differing sign. This is precisely analogous to the baroclinic instability result obtained for two-layer quasi-geostrophic flow (Pedlosky 1987, §7.10) in which instability is possible when the potential vorticity gradients in the two layers are of opposite sign.

It was shown in §3*b*, that in the barotropic limit (i.e. $U_0 \equiv p \equiv 0$), along-shore steady-solutions are linearly stable. In the context of the normal modes this means, of course, that $c_I \equiv 0$. It is straightforward to show that this result indeed follows from (3.13). The real and imaginary parts of (3.13) in the barotropic limit are given by, respectively,

$$\int_0^D h_0(h_{0y})^2[|F_y|^2 + k^2|F|^2] + c_R h_{0y}|F|^2 \, dy = 0, \tag{3.15a}$$

$$c_I \left\{ \int_0^D h_{0y}|F|^2 \, dy \right\} = 0. \tag{3.15b}$$

Assuming instability occurs, it follows from (3.15*b*) that

$$\int_0^D h_{0y}|F|^2 \, dy \equiv 0,$$

which when substituted into (3.15*a*) implies

$$\int_0^D h_0(h_{0y})^2[|F_y|^2 + k^2|F|^2] \, dy \equiv 0,$$

which is clearly only satisfied if $F \equiv 0$ everywhere (the case $h_0 \equiv 0$ being physically uninteresting). Consequently we conclude that there can be no non-trivial unstable solutions to the barotropic problem.

It follows from (3.15*a*) that the real phase for these stable barotropic modes satisfies

$$c_R \equiv -\int_0^D h_0(h_{0y})^2(|F_y|^2 + k^2|F|^2) \, dy \Big/ \int_0^D h_{0y}|F|^2 \, dy. \tag{3.16}$$

Hence the sign of c_R is determined by the sign of the denominator in (3.16). In particular, for a monotonic frontal profile for which $h_{0y} > 0$, this means $c_R < 0$. For example, the linear barotropic stability problem for the 'wedge' front

$$h_0(y) = \alpha y, \tag{3.17}$$

where $\alpha > 0$ and $y \in (0, \infty)$ was solved by Cushman-Roisin (1986). The solutions for $h(y)$ are proportional to $\exp(-ky) L_n(2ky)$ with the dispersion relation

$$c_R = -(2n+1)\,\alpha^2 k, \tag{3.18}$$

where $L_n(*)$ is a Laguerre polynomial of integer order $n \geqslant 0$.

(e) *Matching conditions for $h(y)$ when h_{0y} and/or h_{0yy} are discontinuous*

In practice there are going to be very few smoothly varying frontal profiles $h_0(y)$ for which the stability problem (3.8) can be solved exactly owing to the fact that h_0 appears quadratically in many places in the equations. One way of reducing the problem to a more tractable level is to examine the stability characteristics for a substantially idealized frontal shape in which h_0 is modelled with a completely continuous profile which has regions of differing slopes. Similar procedures have been used frequently in geophysical fluid dynamics to study the stability of various flows (see, for example, LeBlond & Mysak 1978, ch. 7; Drazin & Reid 1981). The solution to the normal mode equations is obtained in each individual region where h_{0y} and h_{0yy} are both continuous and then the individual solutions are 'patched' together using jump conditions which express the continuity of pressure and normal mass flux. In this section we derive these continuity conditions. These constraints will also be useful in determining the required regularity conditions on h and h_y at the outcropping where $h_0 = 0$.

Suppose that along the line $y = \gamma$ that h_{0y} or h_{0yy} possess a finite discontinuity. The condition for pressure continuity in the frontal layer can be derived by integrating (3.8*a*) over $(\gamma-\epsilon, \gamma+\epsilon)$ with respect to y and letting $\epsilon \rightarrow 0^+$. From (3.8*a*) we obtain (after integrating by parts)

$$(h_0 h_{0y} h_y - h_0 h_{0yy} h)\big|_{\gamma-\epsilon}^{\gamma+\epsilon} + \int_{\gamma-\epsilon}^{\gamma+\epsilon} h_{0y}\, p - (k^2 h_0 h_{0y} + c - U_0)\, h \,\mathrm{d}y = 0. \tag{3.19}$$

Assuming that the integrand in (3.19) is bounded will imply that, as $\epsilon \rightarrow 0^+$, the contribution from the integral vanishes and thus pressure continuity is equivalent to

$$[h_0 h_{0y} h_y - h_0 h_{0yy} h] = 0 \tag{3.20}$$

on $y = \gamma$, i.e. the quantity $h_0(h_{0y} h_y - h_{0yy} h)$ is continuous across $y = \gamma$.

To obtain a condition for mass flux continuity, we modify an argument presented in LeBlond & Mysak (1978, §45) for barotropic instability jump conditions. We replace the upper integration limits in (3.19) with y. The result can be written in the form

$$h_0 h_{0y} h_y - h_0 h_{0yy} h - \mathscr{F}(\epsilon) + \int_{\gamma-\epsilon}^{y} h_{0y}\, p - (k^2 h_0 h_{0y} + c - U_0)\, h \,\mathrm{d}y, \tag{3.21}$$

where

$$\mathscr{F}(\epsilon) = h_0(h_{0y} h_y - h_{0yy} h)\big|_{y=\gamma-\epsilon}.$$

Because each individual term is assumed bounded it follows $|\mathscr{F}(\epsilon)| < \infty$. If (3.21) is multiplied through by $(h_{0y})^{-2}$ the result can be put into the form

$$h_0 \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{h}{h_{0y}}\right] - \mathscr{F}(\epsilon)\,(h_{0y})^{-2} + (h_{0y})^{-2} \int_{\gamma-\epsilon}^{y} h_{0y}\, p - (k^2 h_0 h_{0y} + c - U_0)\, h \,\mathrm{d}y. \tag{3.22}$$

If (3.22) is now integrated over $(\gamma-\delta, \gamma+\delta)$, and the double limit $\epsilon, \delta \to 0$ taken, it follows that the integrals associated with the last two terms and the final integral obtained from the integration by parts of the first term vanish due to the assumed boundedness of the integrands. From the first term we obtain simply

$$[hh_0/h_{0y}] = 0, \tag{3.23}$$

on $y = \gamma$. It is possible to interpret the continuity conditions (3.20) and (3.23) as 'low-frequency' limits of (3.9*e*) and (3.9*f*), respectively, where the role of h_0 can be interpreted as acting like a 'density' similar to how the background density appears in the continuity conditions for the Taylor–Goldstein equation (see Drazin & Reid 1981).

In the case where h_0 is continuous across and non-zero on $y = \gamma$, then (3.20) and (3.23) reduce to simply

$$[h_{0y} h_y - h_{0yy} h] = 0, \quad [h/h_{0y}] = 0, \tag{3.24a, b}$$

respectively. On an outcrop, where formally $h_0 = 0$, and h_{0y} and h_{0yy} are clearly discontinuous, (3.20) and (3.23) should be interpreted as

$$\lim_{y\to\gamma} h_0(h_{0y} h_y - h_{0yy} h) = 0, \quad \lim_{y\to\gamma} \frac{h_0 h}{h_{0y}} = 0, \tag{2.25a, b}$$

where the limit path is understood to be in the frontal region. The conditions (3.25*a*, *b*) places certain regularity constraints on $h(y)$ and $h_y(y)$ at an outcrop. These constraints will certainly be satisfied if we insist that h and h_y remain bounded functions.

4. Linear and nonlinear stability of arbitrary steady solutions

(*a*) *Formal stability*

We can use the hamiltonian theory developed in §3 to generate stability conditions for arbitrary steady solutions to (2.24) and (2.25). It is well known (see, for example, Holm *et al.* 1985) that establishing sufficient conditions on the Casimir $C(p, h)$ (2.40) so that

$$\delta\tilde{H}(p_0, h_0) = 0, \tag{4.1}$$

$$\delta^2\tilde{H}(p_0, h_0) \quad \text{is definite,} \tag{4.2}$$

for all perturbations δh and δp where $p_0(x, y)$ and $h_0(x, y)$ are the general steady solutions defined through (2.41) and (2.42) proves the linear stability in the sense of Liapunov (thus excluding both algebraic and modal instability). Holm *et al.* (1985) have termed this sense of stability *formal stability*.

However, unlike in finite-dimensional, the conditions (4.1) and (4.2) are not sufficient to prove *nonlinear* stability because in infinite-dimensions (4.1) and (4.2) are not sufficient to establish that $\tilde{H}(p, h)$ is strictly convex in an open neighbourhood of (p_0, h_0) in the relevant phase space. This is a topological result which is closely related to the fact that in Hilbert space the unit sphere is not compact. (In fact it is this technical issue which Arnol'd (1969) corrects as compared with the stability result established in Arnol'd (1965) for plane flow; for further discussion see Ebin & Marsden (1970) and Ball & Marsden (1984).)

To establish formal stability we examine the second variation of $\tilde{H}$ given by (2.43) where the Casimir C is determined by (2.46) in order that the first order necessary condition (4.1) is satisfied for the general steady solutions defined by (2.41) and (2.42). We find that $\delta^2\tilde{H}(p_0, h_0)$ is given by

$$\delta^2\tilde{H}(p_0,h_0) = \iint_R \nabla(\delta p)\cdot\nabla(\delta p) + F'_{10}(\Delta\delta p+\delta h)^2\,\mathrm{d}x\,\mathrm{d}y$$

$$+\iint_F (\Delta h_0 - F'_{20})\,(\delta h)^2 - h_0\,\nabla(\delta h)\cdot\nabla(\delta h)\,\mathrm{d}x\,\mathrm{d}y, \quad (4.3)$$

where $F'_{10} \equiv \mathrm{d}F_1(\Delta p + h - sy)/\mathrm{d}(\Delta p + h - sy)$ and $F'_{20} \equiv \mathrm{d}F_2(h)/\mathrm{d}h$ evaluated for $(p, h) = (p_0, h_0)$ determined by (2.41) and (2.42). Note that we have reintroduced the notation where R is the spatial region occupied by layer 2 and F is the spatial region occupied by layer 1 as defined in §2*c*. Tedious but straightforward computation shows that $\delta^2\tilde{H}(p_0, h_0)$ is conserved by the linearized dynamics associated with (2.24) and (2.25) given by

$$(\Delta\delta p+\delta h)_t + J[\Delta p_0 + h_0 - sy, F'_{10}(\Delta\delta p+\delta h) - \delta p] = 0,$$

$$\delta h_t + J(\delta p + \delta h\,\Delta h_0 + h_0\,\Delta\delta h + \nabla h_0\cdot\nabla\,\delta h - F'_{20}\,\delta h, h_0) = 0.$$

Formal stability is established if appropriate conditions on the steady state solutions can be found ensuring that $\delta^2\tilde{H}(p_0, h_0)$ is negative or positive definite for all suitably-smooth perturbations δp and δh satisfying the appropriate boundary conditions. It turns out that, in general, conditions cannot be found on F'_{10} and F'_{20} that will guarantee that $\delta^2\tilde{H}(p_0, h_0) > 0$ due to some elementary functional analysis. The argument is as follows. To show positive definiteness ultimately one needs an inequality of the form

$$\iint_F h_0\,\nabla(\delta h)\cdot\nabla(\delta h)\,\mathrm{d}x\,\mathrm{d}y \leqslant \tilde{C}\iint_F (\delta h)^2\,\mathrm{d}x\,\mathrm{d}y,$$

for some positive finite constant $\tilde{C}$ in order to bound the second term in the integrand of the second integral in (4.3) by the first term. However, no finite constant exists since the operator $\hat{L}$ defined through

$$\hat{L}\varphi \equiv -\nabla\cdot(h_0\,\nabla\varphi),$$

is positive, self-adjoint and unbounded (since $h_0 > 0$ in the interior of F). Consequently, the eigenvalues of $\hat{L}$ form a positive real sequence whose only limit point is at infinity (for φ functions which satisfy appropriate boundary conditions on ∂F; namely $\phi(\partial F) = 0$ if $h_0(\partial F) \neq 0$ or $|\phi(\partial F)| < \infty$ if $h_0(\partial F) = 0$; see Zauderer (1989)).

In fact, one can show rather easily that except for a contribution associated with $p_0(x, y)$ the second integral in (4.3) is negative-definite for all δh. If (2.42) is differentiated with respect to y, one obtains

$$-U_0 + h_{0y}\,\Delta h_0 + h_0\,\Delta h_{0y} + \nabla h_0\cdot\nabla h_{0y} = F'_{20}\,h_{0y}, \quad (4.4)$$

which we note is valid for general $U_0 = U_0(x, y)$ and $h_0 = h_0(x, y)$. Using this expression to eliminate F'_{20} allows the second integral in (4.3) to be rewritten in the form

$$\iint_F [U_0/h_{0y} - h_0\,\Delta h_{0y}/h_{0y} - \nabla h_0\cdot\nabla h_{0y}/h_{0y}]\,(\delta h)^2 - h_0\,\nabla(\delta h)\cdot\nabla(\delta h)\,\mathrm{d}x\,\mathrm{d}y,$$

which can be further rewritten in the form

$$\iint_F (U_0/h_{0y})(\delta h)^2 - h_0(h_{0y})^2 \nabla(\delta h/h_{0y})\cdot\nabla(\delta h/h_{0y})\,\mathrm{d}x\,\mathrm{d}y.$$

Thus an alternate representation for $\delta^2\tilde{H}(p_0, h_0)$ is given by

$$\delta^2\tilde{H}(p_0,h_0) = \iint_R \nabla(\delta p)\cdot\nabla(\delta p) + F'_{10}(\Delta\delta p+\delta h)^2\,\mathrm{d}x\,\mathrm{d}y + \iint_F (U_0/h_{0y})(\delta h)^2 - h_0(h_{0y})^2\nabla(\delta h/h_{0y})\cdot\nabla(\delta h/h_{0y})\,\mathrm{d}x\,\mathrm{d}y. \quad (4.5)$$

Notice how the second integral in (4.5) is negative definite except for the U_0/h_{0y} term. The fact that we cannot determine conditions that imply that $\delta^2\tilde{H}(p_0, h_0)$ is *positive* definite is unfortunate because it means that we have been unable to establish the formal stability analogue of Arnol'd's first theorem (Arnol'd 1965).

The determination of conditions to ensure the *negative* definiteness of $\delta^2\tilde{H}(p_0, h_0)$ are most easily obtained if (4.3) is rewritten using the identity

$$\begin{aligned} &F'_{10}(\Delta\delta p+\delta h)^2 + (\Delta h_0 - F'_{20})(\delta h)^2 \\ &\quad\equiv (F'_{10}+\Delta h_0 - F'_{20})[F'_{10}(F'_{10}+\Delta h_0-F'_{20})^{-1}\Delta\delta p+\delta h]^2 \\ &\qquad + F'_{10}(\Delta h_0 - F'_{20})(F'_{10}+\Delta h_0 - F'_{20})^{-1}(\Delta\delta p)^2. \end{aligned} \quad (4.6)$$

Substitution of (4.6) into (4.3) leads to

$$\begin{aligned}\delta^2\tilde{H}(p_0,h_0) = &\iint_R \nabla(\delta p)\cdot\nabla(\delta p) + F'_{10}(\Delta h_0 - F'_{20})(F'_{10}+\Delta h_0-F'_{20})^{-1}(\Delta\delta p)^2 \\ &+ (F'_{10}+\Delta h_0-F'_{20})[F'_{10}(F'_{10}+\Delta h_0 - F'_{20})^{-1}\Delta\delta p+\delta h]^2\,\mathrm{d}x\,\mathrm{d}y \\ &-\iint_F h_0\nabla(\delta h)\cdot\nabla(\delta h)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \quad (4.7)$$

where it is understood that $\delta h \neq 0$ only in region F in the first integral.

To obtain negative-definiteness we need the *Poincaré inequality* (see, for example, Arnol'd 1965; Ladyzhenskaya 1969)

$$\iint_R \nabla(\delta p)\cdot\nabla(\delta p)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{\lambda_{\min}}\iint_R (\Delta\delta p)^2\,\mathrm{d}x\,\mathrm{d}y, \quad (4.8)$$

where $\lambda_{\min}$ is the minimum *positive* eigenvalue of the Sturm–Liouville problem

$$-\Delta\phi = \lambda\phi, \quad (x,y)\in R, \quad (4.9a)$$

$$\phi(\partial R) = 0. \quad (4.9b)$$

So that $\lambda_{\min}$ can be bounded away from zero it is necessary that R be bounded in at least one direction in $\mathbb{R}^2$ (e.g. a channel). Thus for the half-plane domain $R = \{(x,y): 0<y<\infty, -\infty<x<\infty\}$ it follows $\lambda_{\min} = 0$, and (4.8) is of no use. (Other stability theory problems where a Poincaré inequality is needed include Drazin and Reid's (1981, §22) demonstration of a neutrally stable eigenfunction for the Rayleigh problem and many semicircle theorems for barotropic and baroclinic instability (see, for example, Pedlosky 1987, §7.5).)

Substitution of (4.8) in (4.7) gives

$$\delta^2\tilde{H}(p_0,h_0) \leqslant \iint_R [\lambda_{\min}^{-1} + F'_{10}(\Delta h_0 - F'_{20})(F'_{10}+\Delta h_0 - F'_{20})^{-1}](\Delta\delta p)^2 + (F'_{10}+\Delta h_0 - F'_{20})[F'_{10}(F'_{10}+\Delta h_0 - F'_{20})^{-1}\Delta\delta p + \delta h]^2\,\mathrm{d}x\,\mathrm{d}y - \iint_F h_0 \nabla(\delta h)\cdot\nabla(\delta h)\,\mathrm{d}x\,\mathrm{d}y. \quad (4.10)$$

Thus we have the following.

Theorem 4.1. *Steady solutions* $h_0(x,y)$ *and* $p_0(x,y)$ *as determined by* (2.41) *and* (2.42) *are linearly stable in the sense of Liapunov if*

$$F'_{10}(\Delta h_0 - F'_{20})(F'_{10}+\Delta h_0 - F'_{20})^{-1} \leqslant -\lambda_{\min}^{-1}, \quad (4.11a)$$

$$F'_{10}+\Delta h_0 - F'_{20} < 0, \quad (4.11b)$$

for all $(x,y)\in R$ *or* F *as appropriate, where* $\lambda_{\min}$ *is the minimum positive eigenvalue for* (4.9), *and where the disturbance norm denoted* $\|\delta \boldsymbol{q}\|$, *where* $\delta\boldsymbol{q} \equiv (\Delta\delta p + \delta h, \delta h)^{\mathrm{T}}$, *is given by*

$$\|\delta\boldsymbol{q}\|^2 = \iint_R (\Delta\delta p)^2 + (\Delta\delta p + \delta h)^2\,\mathrm{d}x\,\mathrm{d}y.$$

Clearly, the conditions (4.11 a,b) are sufficient to ensure that the right-hand side of (4.10) is negative definite. All that is required now is to show that if the stability conditions (4.11) hold, it is possible to bound the disturbance norm. We begin by noting that it follows from (4.10) and (4.11) that

$$\delta^2\tilde{H}(p_0,h_0) \leqslant \Gamma \iint_R (\Delta\delta p)^2 + (\sigma\Delta\delta p + \delta h)^2\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$\Gamma \equiv \max\{\sup_F [\lambda_{\min}^{-1} + F'_{10}(\Delta h_0 - F'_{20})(F'_{10}+\Delta h_0 - F'_{20})^{-1}], \sup_F (F'_{10}+\Delta h_0 - F'_{20})\} < 0,$$

$$\sigma(x,y) \equiv F'_{10}(F'_{10}+\Delta h_0 - F'_{20})^{-1}.$$

This inequality can be re-arranged to give

$$\iint_R (\Delta\delta p)^2 + (\sigma\Delta\delta p + \delta h)^2\,\mathrm{d}x\,\mathrm{d}y \leqslant \Gamma^{-1}\delta^2\tilde{H}(p_0,h_0).$$

However, it also follows that

$$\|\delta\boldsymbol{q}\|^2 \equiv \iint_R (\Delta\delta p)^2 + [(1-\sigma)(\Delta\delta p) + (\sigma\Delta\delta p + \delta h)]^2\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \iint_R (\Delta\delta p)^2 + 2[(1-\sigma)^2(\Delta\delta p)^2 + (\sigma\Delta\delta p + \delta h)^2]\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \tilde{\sigma} \iint_R (\Delta\delta p)^2 + (\sigma\Delta\delta p + \delta h)^2\,\mathrm{d}x\,\mathrm{d}y,$$

where $\tilde{\sigma} \equiv \max\{2, \sup_F [1+2(1-\sigma)^2]\} > 0$. It therefore follows that

$$\|\delta\boldsymbol{q}\|^2 \leqslant \tilde{\sigma}\Gamma^{-1}\delta^2\tilde{H}(p_0,h_0) \equiv \tilde{\sigma}\Gamma^{-1}\delta^2\tilde{H}(p_0,h_0)|_{t=0},$$

which provides an *a priori* bound on the disturbance norm under the assumption that the stability conditions hold. We can obtain a somewhat more physically relevent estimate if (4.3) is used for $\delta^2\tilde{H}(p_0, h_0)|_{t=0}$. It therefore follows that

$$\|\delta\boldsymbol{q}\|^2 \leqslant \tilde{\sigma}\Gamma^{-1}\left\{\iint_R \nabla(\delta\tilde{p})\cdot\nabla(\delta\tilde{p}) + F'_{10}(\Delta\delta\tilde{p}+\delta\tilde{h})^2 \right.$$

$$+\iint_R (\Delta h_0 - F'_{20})(\delta\tilde{h})^2 - h_0\nabla(\delta\tilde{h})\cdot\nabla(\delta\tilde{h})\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \tilde{\sigma}\Gamma^{-1}\tilde{\Gamma}\left\{\iint_R (\Delta\delta\tilde{p}+\delta\tilde{h})^2\,\mathrm{d}x\,\mathrm{d}y\right.$$

$$\left.+\iint_R (\delta\tilde{h})^2 + \nabla(\delta\tilde{h})\cdot\nabla(\delta\tilde{h})\,\mathrm{d}x\,\mathrm{d}y\right\},$$

where $\delta\tilde{p}(x,y) \equiv \delta p(x,y,0)$, $\delta\tilde{h}(x,y) \equiv \delta h(x,y,0)$ and

$$\tilde{\Gamma} \equiv \min\{\inf_F(F'_{10}), \inf_F(\Delta h_0 - F'_{20}), \inf_F(-h_0)\} < 0,$$

which provides an *a priori* estimate on the disturbance norm in terms of an energy/potential enstrophy norm on the initial perturbations.

We can obtain slightly easier to interpret stability results in terms of the along shelf velocities. If (2.41) is differentiated with respect to y, one obtains

$$F'_{10} = U_0/(s - h_{0y} + \Delta U_0), \tag{4.12}$$

which is valid for all $U_0 = U_0(x,y)$ and $h_0 = h_0(x,y)$. Substituting (4.12) into (4.5) allows $\delta^2\tilde{H}(p_0, h_0)$ to be expressed in the form

$$\delta^2\tilde{H}(p_0,h_0) = \iint_R \nabla(\delta p)\cdot\nabla(\delta p) + [U_0/(s-h_{0y}+\Delta U_0)]\,(\Delta\delta p + \delta h)^2\,\mathrm{d}x\,\mathrm{d}y$$

$$+\iint_F (U_0/h_{0y})\,(\delta h)^2 - h_0(h_{0y})^2\,\nabla(\delta h/h_{0y})\cdot\nabla(\delta h/h_{0y})\,\mathrm{d}x\,\mathrm{d}y. \tag{4.13}$$

If we use the identity

$$[U_0/(s-h_{0y}+\Delta U_0)]\,(\Delta\delta p+\delta h)^2 + (U_0/h_{0y})\,(\delta h)^2$$

$$\equiv \frac{U_0(s+\Delta U_0)}{h_{0y}(s-h_{0y}+\Delta U_0)}[h_{0y}(s+\Delta U_0)^{-1}\,\Delta\delta p + \delta h]^2 + U_0(s+\Delta U_0)^{-1}(\Delta\delta p)^2,$$

then (4.13) can be rewritten in the form

$$\delta^2\tilde{H}(p_0,h_0) = \iint_R \nabla(\delta p)\cdot\nabla(\delta p) + U_0[s+\Delta U_0]^{-1}(\Delta\delta p)^2$$

$$+U_0(s+\Delta U_0)\,[h_{0y}(s-h_{0y}+\Delta U_0)]^{-1}\,[h_{0y}(s+\Delta U_0)^{-1}\Delta\delta p+\delta h]^2\,\mathrm{d}x\,\mathrm{d}y$$

$$-\iint_F h_0(h_{0y})^2\nabla(\delta h/h_{0y})\cdot\nabla(\delta h/h_{0y})\,\mathrm{d}x\,\mathrm{d}y. \tag{4.14a}$$

Substitution of the Poincaré inequality (4.8) into (4.14a) implies

$$\delta^2\tilde{H}(p_0,h_0) \leqslant \iint_R \{\lambda_{\min}^{-1}+\mathrm{U}_0[\mathrm{s}+\Delta U_0]^{-1}\}(\Delta\delta p)^2$$
$$+U_0(s+\Delta U_0)\,[h_{0y}(s-h_{0y}+\Delta U_0)]^{-1}\,[h_{0y}(s+\Delta U_0)^{-1}\,\Delta\delta p+\delta h]^2\,\mathrm{d}x\,\mathrm{d}y$$
$$-\iint_F h_0(h_{0y})^2\nabla(\delta h/h_{0y})\cdot\nabla(\delta h/h_{0y})\,\mathrm{d}x\,\mathrm{d}y. \qquad (4.14b)$$

Conditions for formal stability are therefore given by the following.

Theorem 4.2. *Steady solutions $h_0(x,y)$ and $p_0(x,y)$ as determined by* (2.41) *and* (2.42) *are linearly stable in the sense of Liapunov if*

$$U_0/(s+\Delta U_0) \leqslant -\lambda_{\min}^{-1}, \quad h_{0y}(s-h_{0y}+\Delta U_0) > 0, \qquad (4.15a,b)$$

for all $(x,y)\in R$ or F as appropriate, where $\lambda_{\min}$ is the minimum positive eigenvalue associated with (4.9) *and the disturbance norm is given in Theorem* 4.1.

It is important to note that the stability conditions (4.11) and (4.15) explicitly require the existence of *sheared* flow in the shelf water, i.e. $\nabla U_0 \neq 0$. If U_0 is constant, then by the galilean invariance of the governing equations we may set $U_0 \equiv 0$ or equivalently $F'_{10} \equiv 0$. Clearly, in this situation, it will be impossible to satisfy (4.11a) or (4.15a).

The normal mode stability Theorem 3.1 and Theorem 3.2 results given in §3d for *along-shelf* steady solutions can be seen to be special cases of Theorem 4.2. Inequality (4.15b) implies that either

$$h_{0y} > 0, \quad 0 < h_{0y} < \Delta U_0+s, \qquad (4.16a,b)$$

or

$$h_{0y} < 0, \quad \Delta U_0+s < h_{0y} < 0, \qquad (4.17a,b)$$

must hold. The inequalities (4.16) and (4.17) restricted to the special case $h_0 = h_0(y)$ and $U_0 = U_0(y)$ are exactly Theorems 3.1 and 3.2 respectively.

(*b*) *The barotropic limit*

The barotropic limit for the original model equations (2.24) and (2.25) corresponds to setting $p = 0$ in (2.25) and ignoring (2.24) altogether. The general steady solutions (2.41) and (2.42) reduce to simply

$$h_0\,\Delta h_0+\tfrac{1}{2}\nabla h_0\cdot\nabla h_0 = F_2(h_0). \qquad (4.18)$$

The hamiltonian structure remains, albeit with the obvious modifications, and the second variation of the constrained (reduced) hamiltonian will be given by

$$\delta^2\tilde{H}(h_0) = \iint_F (\Delta h_0-F'_{20})\,(\delta h)^2-h_0\,\nabla(\delta h)\cdot\nabla(\delta h)\,\mathrm{d}x\,\mathrm{d}y, \qquad (4.19a)$$

which is just (4.3) with $\delta p = F'_{10} \equiv 0$. Following the arguments used to derive (4.5), it follows that in the barotropic limit

$$\delta^2\tilde{H}(h_0) = -\iint_F h_0(h_{0y})^2\,\nabla(\delta h/h_{0y})\cdot\nabla(\delta h/h_{0y})\,\mathrm{d}x\,\mathrm{d}y, \qquad (4.19b)$$

which is negative definite for all physically realistic $h_0(x,y)$. Hence we have proved:

Theorem 4.3. *All steady solutions to the barotropic problem*

$$h_t + J(h\Delta h + \tfrac{1}{2}\nabla h \cdot \nabla h, h) = 0, \tag{4.20}$$

together with appropriate boundary conditions, are linearly stable in the sense of Liapunov, where the disturbance norm is given by $\|h\| \equiv [-\delta^2 \tilde{H}(h_0)]^{\frac{1}{2}}$.

Finally we note that the invariant L introduced in §3*b* to establish the necessity of the baroclinic coupling for instability can be now identified as $-\delta^2 \tilde{H}(h_0)$ evaluated for along-shelf solutions $h_0 = h_0(y)$.

(*c*) *A nonlinear stability theorem for the baroclinic problem*

It is almost unnecessary to remark that while formal stability is an important result to establish it does not, however, prove stability since a demonstration of stability requires a fully nonlinear analysis. As mentioned previously the central technical issue is that in infinite-dimensions the definiteness of $\delta^2 \tilde{H}(p_0, h_0)$ is not sufficient to ensure that $\tilde{H}(p, h)$ is strictly convex in an open neighbourhood of (p_0, h_0) in phase space which is required so that we can conclude that steady solutions form a proper extremum of the appropriately constrained energy hypersurface. As a consequence additional convexity hypotheses are required to prove *nonlinear* stability in the sense of Liapunov. The analysis we present here follows the arguments described in Holm *et al.* (1985).

We begin the development of our nonlinear stability result by introducing the conserved functional

$$\mathscr{L}(p, h) \equiv H(p_0+p, h_0+h) - H(p_0, h_0) + C(p_0+p, h_0+h) - C(p_0, h_0), \tag{4.21}$$

where $H(p, h)$ is the hamiltonian given by (2.32) and $C(p, h)$ is the Casimir given by (2.40) with densities Φ_1 and Φ_2 given by (2.46*a*, *b*), respectively, as defined by (2.41) and (2.42) which themselves formally define the steady state solutions $p_0(x, y)$ and $h_0(x, y)$. The perturbations $p(x, y, t)$ and $h(x, y, t)$ are finite amplitude and $p_{\mathrm{T}} \equiv p_0 + p$ and $h_{\mathrm{T}} \equiv h_0 + h$ are solutions to (2.24) and (2.25) with the appropriate boundary conditions. Consequently, $\mathscr{L}(p, h)$ is an invariant of the full nonlinear dynamics.

Substitution of the explicit representations for H and C into (4.21) yields, after a little algebra,

$$\begin{aligned}\mathscr{L}(p, h) = {} & \frac{1}{2}\iint_R \nabla p \cdot \nabla p \,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\iint_F \Delta h_0\, h^2 - (h_0+h)\,\nabla h \cdot \nabla h \,\mathrm{d}x\,\mathrm{d}y \\ & - \iint_R p_0(\Delta p + h)\,\mathrm{d}x\,\mathrm{d}y + \iint_R \int_{q_0}^{q_0+\Delta p+h} F_1(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y \\ & + \iint_F (\nabla h_0 \cdot \nabla h_0 + h_0\,\Delta h_0 + p_0)\, h \,\mathrm{d}x\,\mathrm{d}y + \iint_F \int_{h_0+h}^{h_0} F_2(\xi)\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4.22}$$

where $q_0 \equiv \Delta p_0 + h_0 - sy$. Substitution of (2.41) and (2.42) into (4.22) leads to

$$\begin{aligned}\mathscr{L}(p, h) = {} & \frac{1}{2}\iint_R \nabla p \cdot \nabla p \,\mathrm{d}x\,\mathrm{d}y + \iint_R \left\{ \int_{q_0}^{q_0+\Delta p+h} F_1(\xi)\,\mathrm{d}\xi - F_1(q_0)\,(\Delta p + h) \right\} \mathrm{d}x\,\mathrm{d}y \\ & + \frac{1}{2}\iint_F \Delta h_0\, h^2 - (h_0+h)\,\nabla h \cdot \nabla h \,\mathrm{d}x\,\mathrm{d}y + \iint_F \left\{ \int_{h_0+h}^{h_0} F_2(\xi)\,\mathrm{d}\xi + F_2(h_0)\, h \right\} \mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.23}$$

Some remarks are in order at this stage. If $\mathscr{L}(p,h)$ is Taylor expanded about $(p, h)=\mathbf{0}$, then to leading order $\mathscr{L}(p,h)=\frac{1}{2}\delta^2\tilde{H}(p_0,h_0)+O(h^3,p^3$, etc.). The higher order terms come from the expressions containing the integrals of F_1 and F_2 and the cubic term with respect to h in the third integral in (4.23). However, observe that the second term in the third integral is negative definite since $(h_0+h)\nabla h\cdot\nabla h \geqslant 0$ because the *total thickness* must satisfy $h_0+h\geqslant 0$ on purely physical grounds. Again, as in our discussion of formal stability, it will not be possible due to operator theory arguments, to bound the second term in the integrand of the third integral by the first term. Consequently, a demonstration of nonlinear stability would appear to necessarily entail an argument which shows the negative definiteness of $\mathscr{L}(p,h)$.

In order to proceed further we need the following convexity hypothesis on F_1 and F_2. Suppose that there exists strictly non-zero real numbers α_1, α_2, β_1 and β_2 for which

$$\alpha_1 < F_1'(\xi) < \beta_1, \quad \alpha_2 < F_2'(\xi) < \beta_2, \qquad (4.24\,a,b)$$

where the prime indicates differentiation with respect to the argument. We will establish further conditions on these constants momentarily. If $(4.24\,a,b)$ are integrated twice, it follows that

$$\tfrac{1}{2}\alpha_1(\Delta p+h)^2 < \int_{q_0}^{q_0+\Delta p+h} F_1(\xi)\,\mathrm{d}\xi - F_1(q_0)\,(\Delta p+h) < \tfrac{1}{2}\beta_1(\Delta p+h)^2, \qquad (4.25\,a)$$

$$-\tfrac{1}{2}\beta_2 h^2 < \int_{h_0+h}^{h} F_2(\xi)\,\mathrm{d}\xi + F_2(h_0)\,h < -\tfrac{1}{2}\alpha_2 h^2, \qquad (4.25\,b)$$

for all q_0, h_0, p and h.

Substitution of $(4.25\,a,b)$ into (4.23) implies that

$$\begin{aligned}
&\frac{1}{2}\iint_R \nabla p\cdot\nabla p+\alpha_1(\Delta p+h)^2\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2}\iint_F(\Delta h_0-\beta_2)\,h^2-(h_0+h)\nabla h\cdot\nabla h\,\mathrm{d}x\,\mathrm{d}y\\
&\quad< \mathscr{L}(p,h) < \frac{1}{2}\iint_R \nabla p\cdot\nabla p+\beta_1(\Delta p+h)^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\frac{1}{2}\iint_F(\Delta h_0-\alpha_2)\,h^2-(h_0+h)\nabla h\cdot\nabla h\,\mathrm{d}x\,\mathrm{d}y\\
&\quad<\frac{1}{2}\iint_R \nabla p\cdot\nabla p+\beta_1(\Delta p+h)^2\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\frac{1}{2}\iint_F(\Delta h_0-\alpha_2)h^2\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\frac{1}{2}\iint_R \nabla p\cdot\nabla p+\beta_1(\Delta h_0-\alpha_2)(\beta_1+\Delta h_0-\alpha_2)^{-1}(\Delta p)^2\\
&\qquad+(\beta_1+\Delta h_0-\alpha_2)\,[\beta_1(\beta_1+\Delta h_0-\alpha_2)^{-1}\Delta p+h]^2\,\mathrm{d}x\,\mathrm{d}y, \qquad (4.26)
\end{aligned}$$

where it is understood that integrands involving h_0 and h are only over the region F. Assuming further that we may introduce the Poincaré inequality

$$\iint_R \nabla p\cdot\nabla p\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{\lambda_{\min}}\iint_R(\Delta p)^2\,\mathrm{d}x\,\mathrm{d}y,$$

where $\lambda_{\min}$ is the minimum eigenvalue associated with (4.9), then

$$\mathscr{L}(p,h) < \frac{1}{2}\iint_R [\lambda_{\min}^{-1} + \beta_1(\Delta h_0 - \alpha_2)(\beta_1 + \Delta h_0 - \alpha_2)^{-1}](\Delta p)^2 + (\beta_1 + \Delta h_0 - \alpha_2)[\beta_1(\beta_1 + \Delta h_0 - \alpha_2)^{-1}\Delta p + h]^2 \,\mathrm{d}x\,\mathrm{d}y. \quad (4.27)$$

Nonlinear stability will be proved when α_2 and β_1 are such that $\mathscr{L}(p,h)$ is negative definite. We can therefore state the following.

Theorem 4.4. *If the functions $F_1(\xi)$ and $F_2(\xi)$, which define the steady-state solutions $p_0(x,y)$ and $h_0(x,y)$ through the relations*

$$p_0 = F_1(\Delta p_0 + h_0 - sy),$$
$$p_0 + h_0\Delta h_0 + \tfrac{1}{2}\nabla h_0 \cdot \nabla h_0 = F_2(h_0),$$

satisfy the convexity estimates

$$\alpha_1 < F_1'(\xi) < \beta_1,$$
$$\alpha_2 < F_2'(\xi) < \beta_2,$$

where α_1, β_1, α_2 and β_2 are strictly non-zero real constants satisfying

$$\beta_1(\Delta h_0 - \alpha_2)(\beta_1 + \Delta h_0 - \alpha_2)^{-1} \leqslant -1/\lambda_{\min}, \quad (4.28a)$$
$$\beta_1 - \alpha_2 + \Delta h_0 < 0, \quad (4.28b)$$

for every point (x,y) in R or F as appropriate, and where $\lambda_{\min}$ is the minimum eigenvalue of the homogeneous Dirichlet problem (4.9), *then the steady solutions $p_0(x,y)$ and $h_0(x,y)$ are nonlinearly stable in the sense of Liapunov with respect to the disturbance norm $\|\boldsymbol{q}\|$ given by*

$$\|\boldsymbol{q}\|^2 \equiv \iint_R (\Delta p)^2 + (\Delta p + h)^2 \,\mathrm{d}x\,\mathrm{d}y. \quad (4.29)$$

All that remains to be shown is the following *a priori* estimate on the disturbance norm. Assuming the (4.28) holds, it follows from (4.27) that

$$\mathscr{L}(p,h) < \Gamma \iint_R (\Delta p)^2 + [\beta_1(\beta_1 + \Delta h_0 - \alpha_2)^{-1}\Delta p + h]^2 \,\mathrm{d}x\,\mathrm{d}y, \quad (4.30)$$

where

$$\Gamma \equiv \tfrac{1}{2}\max\{\sup_F [\lambda_{\min}^{-1} + \beta_1(\Delta h_0 - \alpha_2)(\beta_1 + \Delta h_0 - \alpha_2)^{-1}], \sup_F(\beta_1 + \Delta h_0 - \alpha_2)\} < 0.$$

This inequality can be re-arranged to give

$$\iint_R (\Delta p)^2 + (\sigma\Delta p + h)^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \Gamma^{-1}\mathscr{L}(p,h), \quad (4.31)$$

where $$\sigma(x,y) \equiv \beta_1(\beta_1 + \Delta h_0 - \alpha_2)^{-1}.$$

However, it also follows that

$$\begin{aligned}\|\boldsymbol{q}\|^2 &\equiv \iint_R (\Delta p)^2 + [(1-\sigma)\Delta p + (\sigma\Delta p + h)]^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \iint_R [1 + 2(1-\sigma)^2](\Delta p)^2 + 2(\sigma\Delta p + h)^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \tilde{\sigma}\iint_R (\Delta p)^2 + (\sigma\Delta p + h)^2 \,\mathrm{d}x\,\mathrm{d}y \end{aligned} \quad (4.32)$$

where $\tilde{\sigma} \equiv \max\{2, \sup_F [1+2(1-\sigma^2)]\} > 0$. Inequalities (4.32) and (4.31) together imply

$$\|\boldsymbol{q}\|^2 \leqslant \tilde{\sigma}\Gamma^{-1}\mathscr{L}(p,h) \equiv \tilde{\sigma}\Gamma^{-1}\mathscr{L}(\tilde{p},\tilde{h}), \tag{4.33a}$$

where $\tilde{p}(x,y) \equiv p(x,y,0)$ and $\tilde{h}(x,y) \equiv h(x,y,0)$. However, from (4.26) we may infer that

$$\|\boldsymbol{q}\|^2 \leqslant \tilde{\sigma}\Gamma^{-1}\tilde{\Gamma}\left\{\iint_R (\Delta\tilde{p}+\tilde{h})^2\,\mathrm{d}x\,\mathrm{d}y + \iint_R \tilde{h}^2+(h_0+\tilde{h})\nabla\tilde{h}\cdot\nabla\tilde{h}\,\mathrm{d}x\,\mathrm{d}y\right\}, \tag{4.33b}$$

where

$$\tilde{\Gamma} \equiv \tfrac{1}{2}\min\{\alpha_1, \inf_F(\Delta h_0-\beta_2), -1\} < 0.$$

Inequality (4.33b) gives an *a priori* estimate on the disturbance norm in terms of the initial potential enstrophy and energy of the perturbation provided the conditions of the stability theorem hold.

(*d*) *Nonlinear stability in the barotropic limit*

Unlike the barotropic limit examined in the formal stability problem discussed in §4b, the *nonlinear* stability of steady flows in the barotropic limit cannot be *unconditionally* established. The barotropic limit for $\mathscr{L}$, which we denote $\mathscr{L}(h)$, obtained by setting $p = F_1 \equiv 0$ in (4.23) is given by

$$\mathscr{L}(h) = \frac{1}{2}\iint_F \Delta h_0\,h^2-(h_0+h)\,\nabla h\cdot\nabla h\,\mathrm{d}x\,\mathrm{d}y+\iint_F\left\{\int_{h_0+h^*}^{h_0} F_2(\xi)\,\mathrm{d}\xi+F_2(h_0)\,h\right\}\mathrm{d}x\,\mathrm{d}y. \tag{4.34}$$

This functional is an exact invariant of the full nonlinear dynamics in the barotropic limit. It is straightforward to verify that if $\mathscr{L}(h)$ is Taylor expanded about $h = 0$, that $\mathscr{L}(h) = \frac{1}{2}\delta^2\tilde{H}(h_0)+O(h^3)$, where $\delta^2\tilde{H}(h_0)$ is given by (4.19a). While it was possible to show in §4b that $\delta^2\tilde{H}(h_0) < 0$ for all suitably smooth steady solutions $h_0 = h_0(x,y)$, it is not, in general, possible to show the $\mathscr{L}(h)$ is unconditionally negative definite. It appears that the best result one can obtain is the barotropic limit of Theorem 4.3 given by the following.

Theorem 4.5. *If the function $F_2(\xi)$ which defines the steady solution $h_0(x,y)$ through the relation*

$$h_0\,\Delta h_0+\tfrac{1}{2}\nabla h_0\cdot\nabla h_0 = F_2(h_0),$$

satisfies the convexity estimate

$$\sup_F(\Delta h_0) < \alpha_2 < F_2'(\xi) < \beta_2 < \infty, \tag{4.35}$$

then the steady solution $h_0(x,y)$ is nonlinearly stable for the barotropic limit of the dynamics in the sense of Liapunov with respect to the disturbance norm given by

$$\|h\|^2 \equiv \iint_R h^2\,\mathrm{d}x\,\mathrm{d}y.$$

It is straightforward to obtain an *a priori* estimate on the disturbance norm $\|h\|$ following arguments similar to those used to obtain (4.33). What is more interesting, however, is that we can use Theorem 4.5 to obtain an *a priori* estimate for the disturbance energy in the barotropic limit. Assuming $\alpha_2 < F_2'(\xi) < \beta_2$, it follows from (4.34) that

$$\iint_F (\Delta h_0 - \beta_2)\,\tilde{h} - (\tilde{h}+h_0)\,\nabla\tilde{h}\cdot\nabla\tilde{h}\,\mathrm{d}x\,\mathrm{d}y < 2\mathscr{L}(\tilde{h}) \equiv 2\mathscr{L}(h)$$

$$< \iint_F (\Delta h_0 - \alpha_2)\,h^2 - (h+h_0)\,\nabla h\cdot\nabla h\,\mathrm{d}x\,\mathrm{d}y,$$

where $\tilde{h}(x,y) \equiv h(x,y,t=0)$. However, assuming (4.35) this expression can be rewritten in the form

$$0 < \iint_F (h+h_0)\,\nabla h\cdot\nabla h + (\alpha_2 - \Delta h_0)\,h^2\,\mathrm{d}x\,\mathrm{d}y$$

$$< \iint_F (\tilde{h}+h_0)\,\nabla\tilde{h}\cdot\nabla\tilde{h}\,\mathrm{d}x\,\mathrm{d}y + (\beta_2 - \Delta h_0)\,\tilde{h}^2\,\mathrm{d}x\,\mathrm{d}y,$$

which in turn implies

$$0 < E(h) < \Gamma E(\tilde{h}), \tag{4.36}$$

where

$$E(h) \equiv \iint_F (h+h_0)\,\nabla h\cdot\nabla h + h^2\,\mathrm{d}x\,\mathrm{d}y, \tag{4.37}$$

$$\Gamma \equiv \frac{\max\,[1, \sup_F\,(\beta_2 - \Delta h_0)]}{\min\,[1, \inf_F\,(\alpha_2 - \Delta h_0)]} > 0. \tag{4.38}$$

5. Conclusions

In this article we have attempted to construct and analyse a new model describing the baroclinic dynamics of density-driven currents and fronts over a sloping continental shelf. The model presented here is motivated by the desire to develop a theory in which the frontal dynamics balances the competing influences associated with the release of mean kinetic energy associated with a geostrophically balanced current, the generation of relative vorticity in the surrounding slope water by baroclinic vortex-tube stretching due to the perturbed front, and the rectifying influence of the underlying background vorticity gradient associated with a sloping bottom. All of these dynamical processes are important in attempting to describe the real dynamics of density-driven currents. Fronts and currents of this kind are typical oceanographic features in the coastal regions of the world oceans.

We were able to show that the model developed here, which was obtained in a formal asymptotic expansion of the appropriate two-layer shallow-water equations, possessed a non-canonical hamiltonian formulation. This structure was especially useful in constructing a rigorous mathematical analysis of the general linear and nonlinear stability problem.

A detailed examination of the linear stability problem was given for both arbitrary and the more physically useful along-shelf steady flows. In particular, we were able to show that in the barotropic limit the model predicts linear stability in the sense of Liapunov thus excluding both algebraic and modal instabilities. This is an important result because it serves to underscore the point that the instabilities that our baroclinic model will produce do not correspond to simply baroclinically-modified modes of those previously found, but rather represent a new class of instabilities not described before.

For the normal-mode instability problem associated with along-shelf steady flows, two stability theorems could be obtained which serve to illustrate the stabilizing influence of the sloping bottom. In addition, it was possible to interpret the necessary condition for instability as a potential vorticity constraint which stated that the leading-order potential must increase in the offshore direction, i.e. a zero in the cross-shelf potential vorticity is not required for instability.

We were able to show, exploiting the hamiltonian structure of the governing equations, that general steady-state solutions satisfy the first-order necessary conditions for extremizing a suitably constrained hamiltonian. In addition, we were able to obtain relatively simple conditions for establishing the formal stability and hence linear stability in the sense of Liapunov for arbitrary steady flows. In the along-shelf flow limit, it was shown that these stability results reduced to the results previously obtained via the normal-mode approximation. Finally, motivated by the formal stability results, we were able to establish sufficient conditions for the nonlinear stability of density-driven currents and fronts and provided a rigorous saturation bound in terms of the initial potential enstrophy and energy of the perturbations.

Preparation of this manuscript was supported in part by an Operating Grant awarded by the Natural Sciences and Engineering Research Council of Canada, and by a Science Subvention awarded by the Department of Fisheries and Oceans of Canada.

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Received 24 January 1992; accepted 7 October 1992