CANADIAN APPLIED
MATHEMATICS QUARTERLY
Volume 7, Number 1, Spring 1999

SUB-INERTIAL DYNAMICS OF DENSITY-DRIVEN FLOWS IN A CONTINUOUSLY STRATIFIED FLUID ON A SLOPING BOTTOM.
Part 3. NONLINEAR STABILITY THEORY

FRANCIS J. POULIN AND GORDON E. SWATERS

ABSTRACT. A nonlinear stability theory is developed for the low frequency dynamics of bottom-intensified density-driven flows within a continuously stratified rotating fluid of finite depth with variable bottom topography. These flows form an important component of the meridional heat transport in Earth's oceans, i.e., the climate system.

1. Introduction. The continental shelf regions of the world's oceans provide an important wave and flow guide. In addition to allowing for the along coast propagation of large scale waves such as continental shelf, Poincare and Kelvin waves, among others, the ambient sloping bottom topography permits the along slope flow of large scale bottom-intensified density currents. These flows arise as a balance between the gravity-driven down slope acceleration of a relatively dense water mass sitting directly on a sloping bottom and the Coriolis effect which deflects the motion to the along slope direction (toward the right in the northern hemisphere). To emphasize the underlying dynamics and to differentiate them from their nonrotating counterparts, e.g., [**3**], we refer to these flows as *mesoscale gravity currents.*

The flows associated with the coastal transport of deep and bottom ocean waters are mesoscale gravity currents. Examples include the Denmark Strait Overflow [**19, 2**], Antarctic Bottom Water [**25**], density intrusions in the Adriatic Sea [**27**], deep water exchange in the Strait of Georgia [**10**] and benthic currents along the New England shelf, e.g., [**7**], among many others. These flows form a critical component of the oceanic thermohaline circulation and consequently play a major role in Earth's evolving climate, see, e.g., [**17, 8**].

Direct numerical simulations of these flows, e.g., [**8**], based on the full Navier-Stokes equations, suggest they exhibit considerable time and spatial variability and it has been of interest to try to develop simpler, but nevertheless nontrivial, models which can be used to better understand the dynamics involved. Swaters [**21**] developed a simple two-layer theory describing the evolution of these currents. This model was based on a low-frequency approximation to the rotating shallow water equations, i.e., the time scale of the motion is greater than the period of rotation, in which the leading order dynamics in the upper layer are principally driven by the stretching/compression of vortex tubes.

The mean flow in the gravity current arises primarily due to a balance between the Coriolis stress and the down slope gravitational acceleration associated with a relatively dense water mass sitting directly on a sloping bottom. This model filtered out classical shear-based instabilities and focused on the convective destabilization of density-driven currents on a sloping bottom. That is, the source for the perturbation kinetic energy of the upper layer is the release of the potential energy associated with the lower layer fluid mass "sliding" down the shelf.

While the Swaters [**21**] model has been quite successful in describing many aspects of the observed and numerically simulated dynamics of these flows, e.g., [**7, 6, 24, 9, 23**], it has not been able to describe the observed vertical structure of the velocity field in the overlying fluid. Recently, Poulin and Swaters [**15**] have extended the Swaters [**21**] model to allow for vertical variations in the density and velocity fields. In [**15**], hereafter referred to as Part 1, the new model was derived as a systematic asymptotic reduction of the Navier-Stokes equations and a comprehensive linear stability analysis was presented. In [**16**], hereafter referred to as Part 2, we described coherent and radiating eddy solutions for the model.

The principal purpose of this paper is to develop a mathematical nonlinear stability theory for steady solutions of our model, based on the underlying noncanonical Hamiltonian structure of the partial differential equations. The essential mathematical difficulty in establishing nonlinear stability for steady solutions for models of the form described here has been that the argument requires the introduction of an a priori estimate bounding the perturbation energy norm by the enstrophy (vorticity squared) norm. This amounts to deriving an appropriate Poincaré inequality for the problem which was not previously known. Recently, however, Yongming et al. [**26**] have derived such an estimate for a related problem in atmospheric dynamics. We have been able to apply their methods and derive a suitable estimate and, as a result,

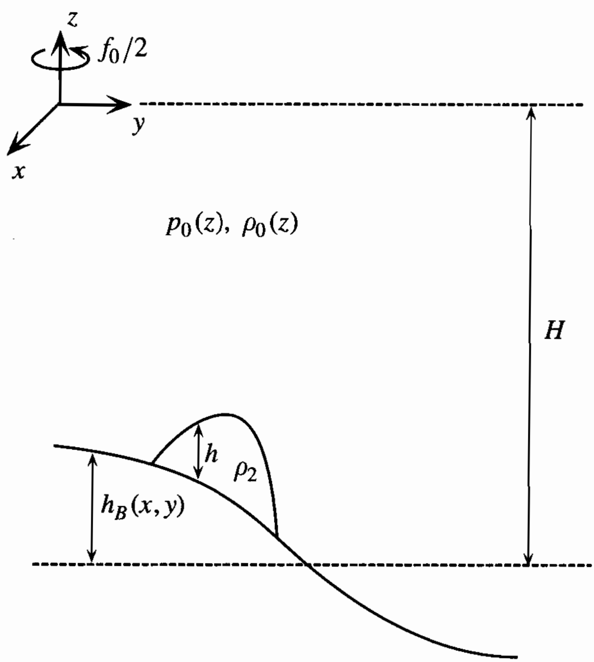

FIGURE 1. Geometry of the model used in this paper.

have been able to establish sufficient conditions for the nonlinear stability, in the sense of Liapunov, of steady solutions to our new model equations.

The plan of this paper is as follows. In Section 2 we briefly describe the governing equations and their interpretation. In Section 3 we describe the noncanonical Hamiltonian structure of the model and present a variational principle for arbitrary steady solutions. In Section 4 we present the stability theory.

2. Governing equations. The underlying geometry and coordinate system is sketched in Figure 1. We assume an incompressible and inviscid two fluid configuration consisting of a stably stratified fluid of finite depth overlying a homogeneous fluid with variable bottom topography in a reference frame rotating with constant angular frequency $f/2$ (f

has units of $\sec^{-1}$). The upper, i.e., the continuously stratified layer, is denoted as layer one. The gravity current, i.e., the lower layer, is denoted as layer two. The upper and lower layer dynamical quantities will be denoted, unless otherwise specified, with a 1 and 2 subscript, respectively.

If the leading order dynamic pressure, i.e., the geostrophic pressure, that is, the total pressure minus the background hydrostatic pressure, in the upper layer is denoted by $\varphi(x, y, z, t)$, the height of the gravity current above the bottom is denoted by $h(x, y, t)$ and the height of the bottom topography relative to the reference height $z = -H$ is denoted by $h_B(x, y)$, the nondimensional model equations can be written in the form (for a detailed derivation, see Part 1)

$$\partial_t[\Delta\varphi + (N^{-2}\varphi_z)_z] + \mu J(\varphi, \Delta\varphi + (N^{-2}\varphi_z)_z) = 0, \tag{2.1}$$

with the dynamic vertical boundary conditions

$$\varphi_{zt} + \mu J(\varphi, \varphi_z) = 0 \quad \text{on} \quad z = 0, \tag{2.2}$$

$$\varphi_{zt} + \mu J(\varphi, \varphi_z) + N^2 J(\varphi + h, h_B) = 0 \quad \text{on} \quad z = -1, \tag{2.3}$$

$$h_t + J(\mu\varphi + h_B, h) = 0 \quad \text{on} \quad z = -1, \tag{2.4}$$

where the Jacobian operator is given by $J(A, B) = A_x B_y - A_y B_x$ and $\Delta = \nabla^2 = \partial_{xx} + \partial_{yy}$. Alphabetical subscripts indicate partial differentiation unless otherwise indicated.

The parameter μ is given by

$$\mu = h_0/(s^* L), \tag{2.5}$$

where s^* is a representative value of the bottom slope $|\nabla h_B|$, L is the horizontal length scale, given by

$$L = \sqrt{g' H}/f,$$

(which is the so-called internal Rossby deformation radius; see, e.g., [**13**]) and H is the reference depth of the entire water column. The nondimensional Brunt-Väisälä frequency, denoted by $N(z)$, is determined by

$$N^2(z) = -\frac{H}{\rho_2 - \rho_0(H)} \left[\frac{d\rho_0(z^*)}{dz^*}\right]_{z^* = Hz} > 0, \tag{2.6}$$

where $\rho_0(z^*)$ is the hydrostatically balanced density profile in the upper layer in the absence of any motion and $\rho_2 > \rho_0(H)$ is the density of the gravity current.

Physically, (2.1) simply expresses the conservation following the flow of the total vorticity in the upper layer where $\Delta\varphi$ is the vertical component of the curl of the velocity field and $(N^{-2}\varphi_z)_z$ is the vorticity associated with the stretching/compression of vortex lines. The parameter μ, which we call the *interaction parameter*, may be interpreted as measuring (as it turns out) the ratio of the destabilizing buoyancy effects to the stabilizing effect of the background vorticity gradient associated with the sloping bottom, see, e.g., [**21**]. The model equations are derived as a systematic low-frequency approximation to the full adiabatic equations governing the flow of a stratified, incompressible, inviscid and rotating fluid. By low frequency we mean that the frequency of the motions we are modeling is small in comparison to the so-called inertial period f^{-1}, i.e., a sub-inertial approximation. Full details of the derivation can be found in Part 1.

The above equations determine the evolution of $\varphi(x, y, z, t)$ and $h(x, y, t)$. Given that these fields are known, the remaining fluid variables are determined by

(2.7) $$\rho = -\varphi_z,$$

(2.8) $$w = -N^{-2}[\varphi_{zt} + \mu J(\varphi, \varphi_z)],$$

(2.9) $$\mathbf{u}_1 = (u_1, v_1) = \mathbf{e}_3 \times \nabla\varphi,$$

(2.10) $$\mathbf{u}_2 = (u_2, v_2) = \mathbf{e}_3 \times \nabla[h_B + \mu(\varphi|_{z=-1} + h)],$$

(2.11) $$p = h_B + \mu(\varphi|_{z=-1} + h),$$

where $\rho(x, y, z, t)$, $w(x, y, z, t)$, $\mathbf{u}_1(x, y, z, t)$, $p(x, y, t)$ and $\mathbf{u}_2(x, y, t)$ are the dynamic density, vertical and horizontal velocities in the upper layer, and the dynamic pressure and horizontal velocity in the lower layer, respectively.

It is necessary to be precise about the spatial domain and additional boundary conditions. The domain, denoted by Ω, given by

(2.12) $$\Omega = \{(x, y, z) \mid z \in (-1, 0), (x, y) \in \Omega_H\},$$

where Ω_H is the simply-connected horizontal component of the domain with smooth boundary denoted $\partial\Omega_H$. Thus (2.1) is solved in the domain Ω for $t > 0$ and (2.2), (2.3) and (2.4) are solved in the domain Ω_H for $t > 0$. From (2.8) we see that (2.2) expresses the physical boundary condition $w = 0$ on $z = 0$, i.e., the upper surface is assumed rigid.

Equations (2.3) and (2.4) together express the *kinematic* condition that a fluid parcel on the deforming interface between the two layers remains on the interface for all time and the *dynamic* condition that the pressure be continuous across the deforming interface. The fact that (2.3) and (2.4) are evaluated on $z = -1$ results from Taylor expanding the full nonlinear boundary conditions on the moving boundary and retaining the leading order, but nevertheless nonlinear, contributions, see Part 1.

From (2.9) we see that φ forms a stream function for the upper layer horizontal velocity $\mathbf{u}_1$. Thus, on the horizontal boundary $\partial\Omega_H$, on which we require the normal flow condition $\mathbf{n}_H \cdot \mathbf{u}_1 = 0$, where $\mathbf{n}_H$ is the unit outward normal, it follows that $\mathbf{t}_H \cdot \nabla\varphi = 0$, where $\mathbf{t}_H$ is a unit tangent vector on $\partial\Omega_H$, for each value of z. It follows that φ must be at most an arbitrary function of (z, t) on $\partial\Omega_H$ which, for convenience, we set to zero. While this is not a necessary assumption in what follows and, indeed, all that we show holds regardless, this choice for the horizontal boundary condition considerably simplifies the derivations. Thus, in summary, we assume

$$\varphi(x, y, z, t) = 0 \quad \text{on} \quad (x, y) \in \partial\Omega_H \quad \forall t \geq 0. \tag{2.13}$$

Similarly, we assume $p =$ constant on $\partial\Omega_H$ for all $t \geq 0$.

3. Hamiltonian structure and variational principle. A system of n partial differential equations written abstractly in the form

$$\Phi\left(\mathbf{q}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}\right) = \mathbf{0}, \tag{3.1}$$

where t is time and $\mathbf{q}(\mathbf{x}, t) = (q_1(\mathbf{x}, t), \dots, q_n(\mathbf{x}, t))^\top$ is a column vector of n dependent variables with the m independent spatial variables $\mathbf{x} = (x_1, \dots, x_m)$ defined on the open spatial domain $\Omega \subset \mathbf{R}^m$ with the boundary (if it exists) $\partial\Omega$, is said to be Hamiltonian, e.g., [**12, 1, 18**] if there exists a conserved functional $H(\mathbf{q})$, called the Hamiltonian, and a matrix $\mathcal{M}$ of (possibly pseudo) differential operators (the cosymplectic form) such that (3.1) can be written in the form

$$\mathbf{q}_t = \mathcal{M}\frac{\delta H}{\delta \mathbf{q}}, \tag{3.2}$$

where $\delta H/\delta q$ is the vector variational or Euler-Lagrange derivative of H with respect to $\mathbf{q}$ and where the Poisson bracket is defined by (see, e.g., [**11**])

$$[F, G] \equiv \left\langle \frac{\delta F}{\delta \mathbf{q}}, \mathcal{M}\frac{\delta G}{\delta \mathbf{q}} \right\rangle, \tag{3.3}$$

where F and G are arbitrary smooth functionals of $\mathbf{q}$ and $\langle *_1, *_2 \rangle$ is the inner product, satisfies the algebraic properties of skew symmetry, distributive and associative laws and the Jacobi identity.

The Hamiltonian formulation is singular if the Poisson bracket is degenerate, that is, if there are nontrivial functionals which satisfy

$$\mathcal{M}\frac{\delta C}{\delta \mathbf{q}} = \mathbf{0}. \tag{3.4}$$

The nontrivial solutions $C = C(\mathbf{q})$ are the time invariant Casimir functionals. These are important since, as it turns out, they are necessary in order to construct a variational principle for arbitrary steady solutions to the model equations.

The Hamiltonian structure for our model is a blend of that for the continuously-stratified quasigeostrophic (QG) equations [**5**] and the two-layer analogue of the present model [**21, 22**]. It was shown in Part 1 that

Theorem 3.1. *The system of equations* (2.1)–(2.4) *is Hamiltonian for the choice of*

$$\begin{aligned} H(\mathbf{q}) = {} & \frac{\mu}{2}\iiint_{\Omega} \nabla\varphi \cdot \nabla + (\varphi_z/N)^2\,dx\,dy\,dz \\ & + \frac{\mu}{2}\iint_{\Omega_H} (h + h_B/\mu)^2 - (h_B/\mu)^2\,dx\,dy, \end{aligned} \tag{3.5}$$

$$\mathbf{q} = (q_1, q_2, q_3, q_4)^\top \tag{3.6}$$

where

$$q_1 = \Delta\varphi + (N^{-2}\varphi_z)_z, \quad q_2 = \varphi_z|_{z=0}, \tag{3.7}$$

$$q_3 = \varphi_z|_{z=-1} + N^2(-1)(h + h_B/\mu), \quad q_4 = h, \tag{3.8}$$

with the Poisson bracket

$$\begin{aligned}[F,G] = &\iiint_\Omega \frac{\delta F}{\delta q_1} J\left(\frac{\delta G}{\delta q_1}, q_1\right) dx\,dy\,dz \\ &- \iint_{\Omega_H} \left[N^2 \frac{\delta F}{\delta q_2} J\left(\frac{\delta G}{\delta q_2}, q_2\right)\right]_{z=0} dx\,dy \\ &+ \iint_{\Omega_H} \left[N^2 \frac{\delta F}{\delta q_3} J\left(\frac{\delta G}{\delta q_3}, q_3\right)\right]_{z=-1} dx\,dy \\ &- \iint_{\Omega_H} \frac{\delta F}{\delta q_4} J\left(\frac{\delta G}{\delta q_4}, q_4\right) dx\,dy.\end{aligned} \tag{3.9}$$

It follows immediately from (3.9) that the cosymplectic form is given by

$$\mathcal{M} = \begin{bmatrix} J(*,q_1) & 0 & 0 & 0 \\ 0 & -N^2(0)J(*,q_2) & 0 & 0 \\ 0 & 0 & N^2(-1)J(*,q_3) & 0 \\ 0 & 0 & 0 & -J(*,q_4) \end{bmatrix}, \tag{3.10}$$

and thus the Casimirs are given by

$$\begin{aligned}C(\mathbf{q}) = &\iiint_\Omega \Phi_1(q_1)\,dx\,dy\,dz \\ &+ \iint_{\Omega_H} \Phi_2(q_2) + \Phi_3(q_3) + \Phi_4(q_4)\,dx\,dy,\end{aligned} \tag{3.11}$$

where Φ_1, Φ_2, Φ_3 and Φ_4 are arbitrary differentiable functions of their arguments.

Let $\varphi_0(x,y,z)$ and $h_0(x,y)$ denote an arbitrary steady state solution of (2.1)–(2.4). It follows that

$$J(\mu\varphi_0, \Delta\varphi_0 + (\varphi_{0_z}/N^2)_z) = 0, \tag{3.12}$$

$$J(\mu\varphi_0, \varphi_{0_z}) = 0 \quad \text{on} \quad z = 0, \tag{3.13}$$

$$J\left(\mu\varphi_0, \varphi_{0_z} + N^2\left(h_0 + \frac{h_B}{\mu}\right)\right) = 0 \quad \text{on} \quad z = -1 \tag{3.14}$$

$$J(\mu\varphi_0 + h_B, h_0) = 0 \quad \text{on} \quad z = -1, \tag{3.15}$$

which in turn imply

$$\mu\varphi_0 = F_1(\Delta\varphi_0 + (\varphi_{0_z}/N^2)_z), \tag{3.16}$$

$$\mu\varphi_0 = F_2(\varphi_{0_z}) \quad \text{on} \quad z = 0, \tag{3.17}$$

$$\mu\varphi_0 = F_3\left(\varphi_{0_z} + N^2\left(h_0 + \frac{h_B}{\mu}\right)\right) \quad \text{on} \quad z = -1, \tag{3.18}$$

$$\mu\varphi_0 + h_B = F_4(h_0) \quad \text{on} \quad z = -1, \tag{3.19}$$

where the F_i are arbitrary functions of their arguments (the possible dependency of F_1 on z has been suppressed).

In Part 1 we established the variational principle

Theorem 3.2. *The steady solutions $\varphi_0(x,y,z)$ and $h_0(x,y)$ given by (3.16)–(3.19) satisfy the first order necessary conditions associated with the constrained Hamiltonian*

$$\mathcal{H} = H + C, \tag{3.20}$$

i.e.,

$$\delta\mathcal{H}(\varphi_0, h_0) = 0, \tag{3.21}$$

provided the Casimir densities are given by

$$\begin{aligned}
&\Phi_1(q_1) = \int_0^{q_1} F_1(\tau)\,d\tau, \quad \Phi_2(q_2) = -N^{-2}(0)\int_0^{q_2} F_2(\tau)\,d\tau, \\
&\Phi_3(q_3) = N^{-2}(-1)\int_{N^2(-1)h_B/\mu}^{q_3} F_3(\tau)\,d\tau, \\
&\Phi_4(q_4) = -\int_0^{q_4} F_4(\tau)\,d\tau - \frac{\mu q_4^2}{2}.
\end{aligned} \tag{3.22}$$

4. Stability theory. The second variation of the constrained Hamiltonian $\mathcal{H}$ evaluated at the steady state solution is

$$\begin{aligned}\delta^2\mathcal{H}(\varphi_0, h_0) = &\iiint_\Omega \mu[\nabla(\delta\varphi)\cdot\nabla(\delta\varphi) + (\delta\varphi_z/N)^2] \\ &+ F'_{10}[\delta q_1]^2\, dx\, dy\, dz \\ &+ \iint_{\Omega_H} F'_{30}[\delta q_3/N]^2_{z=-1} - F'_{20}[\delta q_2/N]^2_{z=0} \\ &- F'_{40}[\delta q_4]^2\, dx\, dy,\end{aligned} \tag{4.1}$$

where $F'_{i0} = dF_i(q_i0)/dq_i$ for $i = 1, 2, 3$ and 4. Henceforth it will be assumed that i ranges from 1 to 4 unless stated otherwise. If $\delta^2\mathcal{H}(\varphi_0, h_0)$ is sign definite for all perturbations, then it can be shown, exploiting the fact that $\delta^2\mathcal{H}(\varphi_0, h_0)$ is an invariant of the linear stability equations (not written here, see Part 1) and a quadratic form with respect to the perturbations, that the mean flow is linearly stable in the sense of Liapunov.

In Part 1 we proved

Theorem 4.1. *Suppose that the Casimir density functions in the variational principle Theorem 3.2 satisfy*

$$(-1)^{i+1}F'_{i0} \geq 0 \quad \textit{for } i = 1, 2, 3, 4, \tag{4.2}$$

then (φ_0, h_0) is linearly stable in the sense of Liapunov with respect to the perturbation norm

$$\|\delta\mathbf{q}\| = [\delta^2\mathcal{H}(\varphi_0, h_0)]^{1/2}. \tag{4.3}$$

This result, which ensures that $\delta^2\mathcal{H}(\varphi_0, h_0)$ is positive definite, is the analogue of Arnol'd's first stability theorem, see, e.g., [**5**] for our model equations. The conditions (4.2) are a straightforward union of the known formal stability results for the continuously-stratified QG equations [**20**] and the two-layer analogue of the present model [**21, 22**]. We also showed in Part 1 how these conditions could be generalized to establish nonlinear stability in the sense of Liapunov by introducing appropriate convexity assumptions on the functionals $\Phi_i(q_i)$

The analogue of Arnol'd's second stability theorem revolves around establishing conditions for the negative definiteness of $\delta^2\mathcal{H}(\varphi_0, h_0)$ for all perturbations. Examining (4.1) leads one to conjecture that $\delta^2\mathcal{H}(\varphi_0, h_0)$ cannot be negative definite for all perturbations unless the $F'_{10}[\delta q_1]^2$ (enstrophy) term is sufficiently negative so as to dominate the positive contribution associated with the $\mu[\nabla(\delta\varphi)\cdot\nabla(\delta\varphi)+(\delta\varphi_z/N)^2]$ (energy) term. Indeed, a crucial aspect of the analysis requires introducing a Poincaré inequality which will bound the upper layer perturbation energy norm by the enstrophy norm. Until recently this estimate was not established. Yongming et al. **[26]** have derived a new Poincaré inequality for the continuously-stratified QG equations. We have been able to exploit this new Poincare inequality to establish the analogue of Arnol'd's second stability theorem for the steady solutions to our model.

Although establishing the analogue of Arnol'd's second stability theorem may seem more of a mathematical curiosity than physically relevant applied mathematics, in fact this is not the case. As it turns out, many of the large scale flows in geophysical fluid dynamics possess the property, qualitatively at least, that

$$F'_{10} = dF_1(q_{10})/dq_1 < 0.$$

That is, considering (3.16), increasing vorticity, i.e., $\Delta\varphi_0 + (\varphi_{0_z}/N^2)_z$, is, on the large scale, associated with decreasing pressure, i.e., φ_0. In the atmosphere, for example, a low pressure region has counterclockwise flow associated with it which has a positive vorticity signature. The same property is true for large scale ocean eddies of the kind one sees in satellite images.

Yongming et al. **[26]** have proven the estimate

$$\begin{aligned}(4.4)\quad &\iiint_\Omega \nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2\,dx\,dy\,dz\\ &\qquad\le \frac{\alpha_1}{K}\iiint_\Omega \delta q_1^2\,dx\,dy\,dz\\ &\qquad\quad + \iint_{\Omega_H} \frac{\alpha_2}{K}(\delta\varphi_z/N)^2_{z=0} + \frac{\alpha_3}{K}(\delta\varphi_z/N)^2_{z=-1}\,dx\,dy,\end{aligned}$$

where α_1, α_2 and α_3 are positive constants and K is the smallest positive solution of

$$(4.5)\qquad \frac{\alpha_1}{K} = \frac{1}{\lambda_{\min}(K)},$$

where $\lambda_{\min}(K) > 0$ is the smallest eigenvalue associated with the problem

$$(\nabla\varphi + (N^{-2}\varphi_z)_z) + \lambda\varphi = 0, \tag{4.6}$$

with the boundary conditions

$$\alpha_2\varphi_z - K\varphi = 0 \quad \text{on} \quad z = 0, \tag{4.7}$$

$$\alpha_3\varphi_z + K\varphi = 0 \quad \text{on} \quad z = -1, \tag{4.8}$$

$$\varphi = 0 \quad \text{on} \quad \partial\Omega_H. \tag{4.9}$$

If (4.4) is substituted into (4.1), it follows that

$$\begin{aligned}
\delta^2\mathcal{H}(\mathbf{q}_0) \leq & \iiint_\Omega \left(\frac{\alpha_1\mu}{K} + F'_{10}\right)(\delta q_1)^2\,dx\,dy\,dz \\
& + \iint_{\Omega_H} \left(\frac{\alpha_2\mu}{K} - F'_{20}\right)(\delta q_2/N)^2_{z=0}\,dx\,dy \\
& + \iint_{\Omega_H} \left(\frac{\alpha_3\mu}{K}(\delta\varphi_z/N)^2_{z=-1}\right. \\
& \qquad \left. + F'_{30}(\delta q_3/N)^2_{z=-1} - F'_{40}(\delta q_4)^2\right)dx\,dy \\
= & \iiint_\Omega \left(\frac{\alpha_1\mu}{K} + F'_{10}\right)(\delta q_1)^2\,dx\,dy\,dz \\
& + \iint_{\Omega_H} \left(\frac{\alpha_2\mu}{K} - F'_{20}\right)(\delta q_2/N)^2_{z=0}\,dx\,dy \\
& + \iint_{\Omega_H} \left(\frac{\alpha_3\mu}{K} + F'_{30}\right)((\delta\varphi_z + \gamma N^2\delta h)/N)^2_{z=-1}\,dx\,dy \\
& + \iint_{\Omega_H} \left(\frac{N^2(-1)\alpha_3\mu\gamma}{K} - F'_{40}\right)(\delta q_4)^2\,dx\,dy,
\end{aligned} \tag{4.10}$$

where

$$\gamma = \frac{F'_{30}}{(\alpha_3\mu/K) + F'_{30}}. \tag{4.11}$$

Theorem 4.2. *The steady solutions $\varphi_0(x,y,z)$ and $h_0(x,y)$, as defined by the variational principle Theorem* 3.2, *are linearly stable in the sense of Liapunov with respect to the perturbation norm*

$$\|\delta \mathbf{q}\| = [-\delta^2 \mathcal{H}(\mathbf{q}_0)]^{1/2}, \tag{4.12}$$

if the Casimir density functions satisfy the conditions

$$(-1)^{i+1} F'_{i0} < -\frac{\alpha_i \mu}{K} \quad \textit{for } i = 1, 2, 3, \tag{4.13}$$

$$-F'_{40} < -N^2(-1)\frac{\alpha_3 \mu \gamma}{K}, \tag{4.14}$$

where γ is given by (4.11).

Proof. Clearly (4.13) and (4.14) are sufficient to ensure that $\delta^2\mathcal{H}(\mathbf{q}_0)$ is negative definite. Thus, exploiting the invariance of $\delta^2\mathcal{H}(\mathbf{q}_0)$, with respect to the linear stability equations, we have

$$0 < \|\delta \mathbf{q}\|^2(t) = -\delta^2\mathcal{H}(\mathbf{q}_0) = [-\delta^2\mathcal{H}(\mathbf{q}_0)]_{t=0} = \|\delta \mathbf{q}\|^2(0). \quad \square$$

Of course, while establishing conditions for the linear stability of steady solutions is interesting, it hardly needs saying that linear stability does not imply nonlinear stability. Nevertheless, the linear results are useful in that they point to the direction one must take to prove nonlinear stability.

Conditions for nonlinear stability can be established using the functional,

$$\mathcal{L}(\delta \mathbf{q}) \equiv \mathcal{H}(\delta \mathbf{q} + \mathbf{q}_0) - \mathcal{H}(\mathbf{q}_0), \tag{4.15}$$

where $\mathcal{H}$ is defined in (3.20), $\mathbf{q}_0$ denotes the steady state solution, $\mathbf{q} = \delta\mathbf{q}+\mathbf{q}_0$ is the total flow field and $\delta\mathbf{q}$ is the finite-amplitude perturbation. We remark that $\mathcal{L}$ is an invariant of the full nonlinear dynamics (2.1)–(2.4) since each individual functional is. It is straightforward to verify that the lead term in the Taylor expansion of $\mathcal{L}(\delta\mathbf{q})$ about

$\delta\mathbf{q} = \mathbf{0}$ is just $\delta^2\mathcal{H}(\mathbf{q}_0)/2$. Substituting $\mathcal{H}$ into $\mathcal{L}$ yields

$$\begin{aligned}\mathcal{L}(\delta\mathbf{q}) = \iiint_\Omega \bigg\{ &\frac{\mu}{2}[\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2] \\ &+ \int_{q_{10}}^{q_{10}+\delta q_1} (F_1(\tau) - F_1(q_{10}))\,d\tau \bigg\}\,dx\,dy\,dz \\ + \iint_{\Omega_H} \bigg\{ &\left[N^{-2}\int_{q_{30}}^{q_{30}+\delta q_3} (F_3(\tau) - F_3(q_{30}))\,d\tau\right]_{z=-1} \\ &- \left[N^{-2}\int_{q_{20}}^{q_{20}+\delta q_2} (F_2(\tau) - F_2(q_{20}))\,d\tau\right]_{z=0} \\ &- \left[\int_{q_{40}}^{q_{40}+\delta q_4} (F_4(\tau) - F_4(q_{40}))\,d\tau\right] \bigg\}\,dx\,dy.\end{aligned} \tag{4.16}$$

It is elementary that if the $F_i(\tau)$ functions satisfy

$$0 < \alpha_i < (-1)^i F_i' < \beta_i < \infty, \tag{4.17}$$

for all values of their argument, then

$$\begin{aligned} -\frac{\beta_i \delta q_i^2}{2} &< (-1)^{i+1}\int^{q_{i0}+\delta q_i} q_{i0}[F_i(\tau) - F_i(q_{i0})]\,d\tau \\ &< -\frac{\alpha_i\delta q_i^2}{2}, \end{aligned} \tag{4.18}$$

for any δq_i. Substituting (4.18) into (4.16) implies

$$\begin{aligned} \frac{1}{2}\iiint_\Omega &\mu[\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2] - \beta_1(\delta q_1)^2\,dx\,dy\,dz \\ &- \frac{1}{2}\int_{\Omega_H} \beta_2(\delta q_2)^2 + \beta_3(\delta q_3)^2 + \beta_4(\delta q_4)^2\,dx\,dy \\ &\leq \mathcal{L} \leq \frac{1}{2}\iiint_\Omega (\mu[\nabla\delta\varphi\cdot\nabla\delta\varphi + (\delta\varphi_z/N)^2] - \alpha_1(\delta q_1)^2)\,dx\,dy\,dz \\ &\qquad - \frac{1}{2}\iint_{\Omega_H} (\alpha_2(\delta q_2)^2 + \alpha_3(\delta q_3)^2 + \alpha_4(\delta q_4)^2)\,dx\,dy. \end{aligned} \tag{4.19}$$

The analogue of Arnol'd's second theorem for the nonlinear stability of steady solutions to (2.1)–(2.4) is

Theorem 4.3. *If the Casimir density functions $F_i(\tau)$ for $i = 1, \ldots, 4$, as given in the variational principle Theorem* 3.2, *satisfy* (4.17) *with*

$$K > \mu, \tag{4.20}$$

$$\alpha_4 > N^2(-1)\frac{\alpha_3 \mu \gamma}{K}, \quad \text{where } \gamma \equiv \frac{K}{K-\mu}, \tag{4.21}$$

then the steady solutions φ_0 and h_0 are nonlinearly stable in the sense of Liapunov with respect to the norm $\|\delta \mathbf{q}\|$ given by

$$\begin{aligned} \|\delta \mathbf{q}\|^2 = & \iiint_\Omega (\delta q_1)^2 \, dx\, dy\, dz \\ & + \iint_{\Omega_H} (\delta q_2/N)^2_{z=0} \, dx\, dy \\ & + \iint_{\Omega_H} (\delta \varphi_z/N)^2_{z=-1} + (\delta q_4)^2 \, dx\, dy. \end{aligned} \tag{4.22}$$

Proof. As with the proof of Theorem 4.2, we require the Poincare inequality (4.4). It follows that

$$\begin{aligned} \mathcal{L} \le & \frac{1}{2}\left(\frac{\mu}{K} - 1\right)\left(\iiint_\Omega \alpha_1 (\delta q_1)^2 \, dx\, dy\, dz \right. \\ & \left. + \iint_{\Omega_H} \alpha_2 (\delta q_2/N)^2_{z=0} \, dx\, dy\right) \\ & + \frac{1}{2}\iint_{\Omega_H} \left(\frac{\alpha_3 \mu}{K}(\delta \varphi_z/N)^2_{z=-1} \right. \\ & \left. - \alpha_3 (\delta q_3/N)^2_{z=-1} - \alpha_4 (\delta q_4)^2\right) dx\, dy \\ = & \frac{1}{2}\left(\frac{\mu}{K} - 1\right)\left(\iiint_\Omega \alpha_1 (\delta q_1)^2 \, dx\, dy\, dz \right. \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&\qquad + \iint_{\Omega_H} \alpha_2 (\delta q_2/N)^2_{z=0}\, dx\, dy \Big) \\
&+ \frac{1}{2}\Big(\frac{\mu}{K} - 1\Big) \iint_{\Omega_H} \alpha_3 (\delta\varphi_z/N + \gamma N \delta h)^2_{z=-1}\, dx\, dy \\
&+ \frac{1}{2} \iint_{\Omega_H} \Big(N^2(-1)\frac{\alpha_3 \mu\gamma}{K} + \alpha_4\Big) (\delta q_4)^2\, dx\, dy \\
&\le \frac{1}{2}\tilde{\Upsilon}^{-1} \Big\{ \iiint_{\Omega} (\delta q_1)^2\, dx\, dy\, dz \\
&\qquad + \iint_{\Omega_H} (\delta q_2/N)^2_{z=0}\, dx\, dy \\
&\qquad + \iint_{\Omega_H} (\delta\varphi_z/N + \gamma N \delta h)^2_{z=-1} + (\delta q_4)^2\, dx\, dy \Big\},
\end{aligned}
$$

where γ is defined by (4.21), which is positive by assumption, and

$$
\tilde{\Upsilon}^{-1} = \max_{i=1,2,3} \Big\{ \alpha_i \Big(\frac{\mu}{K} - 1\Big), \Big(N^2(-1)\frac{\alpha_3 \mu \gamma}{K} + \beta_4\Big) \Big\} < 0. \tag{4.24}
$$

Since $\tilde{\Upsilon}$ is negative, it follows from (4.23) that

$$
\begin{aligned}
&\iiint_{\Omega} (\delta q_1)^2\, dx\, dy\, dz + \iint_{\Omega_H} (\delta q_2/N)^2_{z=0}\, dx\, dy \\
&\qquad + \iint_{\Omega_H} (\delta\varphi_z/N + \gamma N \delta h)^2_{z=-1} + (\delta q_4)^2\, dx\, dy \\
&\qquad\qquad \le 2\tilde{\Upsilon}\mathcal{L} = 2\tilde{\Upsilon}\mathcal{L}_{t=0},
\end{aligned} \tag{4.25}
$$

having used the invariance of $\mathcal{L}$.

As well, the following upper bound can be established

$$
\begin{aligned}
\|\delta \mathbf{q}\|^2 = &\iiint_{\Omega} (\delta q_1)^2\, dx\, dy\, dz \\
&+ \iint_{\Omega_H} (\delta q_2/N)^2_{z=0}\, dx\, dy \\
&+ \iint_{\Omega_H} (\delta\varphi_z/N + \gamma N \delta q_4 - \gamma N \delta q_4)^2_{z=-1} + (\delta q_4)^2\, dx\, dy
\end{aligned} \tag{4.26}
$$

$$\begin{aligned}
&\leq \iiint_{\Omega} (\delta q_1)^2 \, dx \, dy \, dz \\
&\quad + \iint_{\Omega_H} (\delta q_2/N)^2_{z=0} \, dx \, dy \\
&\quad + \iint_{\Omega_H} \Big(2(\delta \varphi_z/N + \gamma N \delta q_4)^2_{z=-1} \\
&\qquad\qquad + (1 + 2\gamma^2 N^2(-1))(\delta q_4)^2 \Big) \, dx \, dy \\
&\leq \tilde{\Upsilon} \Big\{ \iiint_{\Omega} (\delta q_1)^2 \, dx \, dy \, dz \\
&\qquad + \iint_{\Omega_H} (\delta q_2/N)^2_{z=0} \, dx \, dy \\
&\qquad + \iint_{\Omega_H} (\delta \varphi_z/N + \gamma N \delta q_4)^2_{z=-1} + (\delta q_4)^2 \, dx \, dy \Big\},
\end{aligned}$$

where we define

$$\bar{\Upsilon} = \max\{2, 1 + 2\gamma^2 N^2(-1)\} > 0. \tag{4.27}$$

Now substituting (4.25) into (4.26) yields, upon noting that $\bar{\Upsilon}\tilde{\Upsilon} < 0$,

$$\begin{aligned}
\|\delta \mathbf{q}\|^2 &\leq 2\bar{\Upsilon}\tilde{\Upsilon}\mathcal{L}_{t=0} \\
&\leq \bar{\Upsilon}\tilde{\Upsilon} \Big\{ \iiint_{\Omega} \Big(\mu(\nabla \delta\varphi \cdot \nabla \delta\varphi + (\delta\varphi_z/N)^2) \\
&\qquad\qquad - \beta_1(\delta q_1)^2 \Big) \, dx \, dy \, dz \\
&\qquad - \iint_{\Omega_H} \Big(\beta_2(\delta q_2/N)^2_{z=0} + \beta_3(\delta q_3/N)^2_{z=-1} \\
&\qquad\qquad + \beta_4(\delta q_4)^2 \Big) \, dx \, dy \Big\}_{t=0} \\
&\leq -\bar{\Upsilon}\tilde{\Upsilon} \Big\{ \iiint_{\Omega} \beta_1(\delta q_1)^2 \, dx \, dy \, dz \\
&\qquad + \iint_{\Omega_H} \beta_2(\delta q_2/N)^2_{z=0} \, dx \, dy \\
&\qquad + \iint_{\Omega_H} 2\beta_3(\delta\varphi_z/N)^2_{z=-1}
\end{aligned} \tag{4.28}$$

$$+ (2N^2(-1)\beta_3 + \beta_4)(\delta q_4)^2 \, dx \, dy \Big\}_{t=0}$$
$$\leq \Upsilon \|\delta \mathbf{q}\|_{t=0}^2,$$

where we define

$$\Upsilon = \tilde{\Upsilon}\tilde{\Upsilon} \max_{i=1,2} \{\beta_i, 2\beta_3, 2N^2(-1)\beta_3 + \beta_4\} > 0. \tag{4.29}$$

Therefore, for every $\epsilon > 0$ there exists a $\delta = \Upsilon^{-1/2}\epsilon > 0$ such that $\|\delta \mathbf{q}\|_{t=0} < \delta \Rightarrow \|\delta \mathbf{q}\| < \epsilon$ for all $t \geq 0$. □

5. Conclusions. Deep boundary currents in the oceans are the principal mechanism by which cold dense waters formed in high latitude regions flow southward back toward equatorial regions. These currents are principally driven by a balance between pressure gradients and the Coriolis effect and flow along the margins of the oceans on the continental shelves. Relatively little is known about their dynamics due, in part, to the enormous costs of direct oceanographic observations of these deep currents. However, numerical simulations of coupled atmosphere-ocean climate models suggest that these currents are highly variable in space and time. Their intrinsic dynamical properties may make an important dynamical contribution to the internal variability in the earth climate system. We have been attempting to develop simple, but nontrivial, mathematical and computer models in order to understand the dynamics of these currents.

In this paper we developed the mathematical theory associated with the nonlinear stability of these flows. Our stability theory has been based on using the energy-Casimir variational method modified to take into account recent developments in deriving a Poincaré inequality for the continuously-stratified QG equations. Prior to the establishment of this inequality, it was not known how, or if it was possible, to bound the perturbation energy norm by the perturbation enstrophy norm. This was a crucial obstacle preventing the establishment of nonlinear stability within the most physically important context. It is anticipated that the mathematical theory we have developed here will prove useful in other fluid dynamical stability problems.

Acknowledgments. This work was completed while G.E.S. was an Invited Participant in the Mathematics of Atmosphere and Ocean

Dynamics Programme hosted by the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge, U.K. during 1996. Preparation of this manuscript was supported in part by a Research Grant awarded by the Natural Sciences and Engineering Research Council of Canada and by a Science Subvention award by the Department of Fisheries and Oceans of Canada to G.E.S. and by a Province of Alberta Postgraduate scholarship awarded to F.J.P.

REFERENCES

1 T.B. Benjamin, *Impulse flow force and variational principles*, IMA J. Appl. Math. **32**, 3–68.

2. J.G. Bruce, *Eddies southwest of the Denmark Strait*, Deep-Sea Res. **42** (1995), 13–29.

3. R.E. Britter and P.F. Linden, *The motion of the front of a gravity current travelling down an incline*, J. Fluid Mech. **99** (1980), 531–543.

4. D.D. Holm, *Hamiltonian formulation of the baroclinic quasi-geostrophic fluid equations*, Phys. Fluids **29** (1985), 7–8.

5. D.D. Holm, J.E. Marsden, T. Raitu and A. Weinstein, *Nonlinear stability of fluid and plasma equilibria*, Phys. Rep. **123** (1985), 1–116.

6. H. Honji and T. Hosyamada, *Instability of the rotating gravity current flow down a slope*, Rep. Res. Inst. Appl. Mech. **35** (1989), 55–64.

7. R.W. Houghton, R. Schlitz, R.C. Beardsley, B. Butnam and J.C. Chamberlin, *The middle Atlantic bight pool: Evolution of the temperature structure during 1979*, J. Phys. Oceangr. **25** (1982), 1019–1029.

8. L. Jian and R.W. Garwood, Jr., *Three-dimensional simulations of overflows on continental slopes*, J. Phys. Oceanogr. **26** (1996), 1214–1233.

9. R.H. Karsten, G.E. Swaters and R.E. Thompson, *Stability characteristics of deep-water replacement in the Strait of Georgia*, J. Phys. Oceanogr. **25** (1995), 2391–2403.

10. P.H. LeBlond, H. Ma, F. Doherty and S. Pond, *Deep and intermediate water replacement in the Strait of Georgia*, Atmos.-Ocean **29** (1991), 288–312.

11. P.J. Morrison, *Poisson brackets for fluids and plasmas*, in *Mathematical methods in hydrodynamics and integrability in dynamical systems* (M. Tabor and Y.M. Treve, eds.), A.I.P. Conference Proceedings **88** (1982), 13–46.

12. P.J. Olver, *A nonlinear Hamiltonian structure for the Euler equations*, J. Appl. Math. Anal. **89** (1982), 233–250.

13. J. Pedlosky, *Geophysical fluid dynamics*, Springer-Verlag, 1987.

14. F.J. Poulin, *Mesoscale gravity currents and cold-pools within a continuously stratified fluid overlying gently sloping topography*, M.Sc. Thesis, Department of Mathematical Sciences, University of Alberta.

15. F.J. Poulin and G.E. Swaters, *Sub-inertial dynamics of density-driven flows in a continuously stratified fluid on a sloping bottom*, Part 1: *Model formulation and stability characteristics*, Proc. Roy. Soc. London, submitted.

16. ———, *Sub-inertial dynamics of density-driven flows in a continuously stratified fluid on a sloping bottom*, Part 2: *Isolated eddies and radiating cold domes*, Proc. Roy. Soc. London, submitted.

17. J. Price and M.O. Baringer, *Outflows and deep water production by marginal seas*, Prog. Oceanogr. **33** (1994), 161–200.

18. T.G. Shepherd, *Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics*, Adv. Geophys. **32** (1990), 287–335.

19. P.C. Smith, *Baroclinic instability in the Denmark Strait overflow*, J. Phys. Oceanogr. **6** (1976), 355–371.

20. G.E. Swaters, *A nonlinear stability theorem for baroclinic quasi-geostrophic flow*, Phys. Fluids **29** (1985), 5–6.

21. ———, *On the baroclinic instability of cold-core coupled density fronts on a sloping continental shelf*, J. Fluid Mech. **224** (1991), 361–382.

22. ———, *Nonlinear stability of intermediate baroclinic flow on a sloping bottom*, Proc. R. Soc. London **442** (1993), 249–272.

23. ———, *Numerical simulations of the baroclinic dynamics of density-driven coupled fronts and eddies on a sloping bottom*, J. Geophys. Res., in press.

24. G.E. Swaters and G.R. Fierl, *Dynamics of ventilated coherent cold eddies on a sloping bottom*, J. Fluid Mech. **223** (1991), 565–587.

25. J.A. Whitehead and L.U. Worthington, *The flux and mixing rates of Antarctic Bottom Water within the North Atlantic*, J. Geophys. Res. **87** (1982), 7903–7924.

26. L. Yongming, M. Mu and T.G. Shepherd, *Nonlinear stability of continuously stratified quasi-geostrophic flow*, J. Fluid Mech. **325** (1996), 419–439.

27. L. Zoccolotti and E. Salusti, *Observations on a very dense marine water in the Southern Adriatic Sea*, Continent. Shelf. Res. **7** (1987), 535–551.

Applied Mathematics Institute, Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada
E-mail address: gordon.swaters@ualberta.ca