Nonlinear stability of intermediate baroclinic flow on a sloping bottom

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The governing equations describing the dynamics of mesoscale gravity currents or coupled density fronts and steadily-travelling coherent cold eddies on a sloping bottom are shown to possess a non-canonical hamiltonian structure. We exploit the hamiltonian formalism to obtain a variational principle that describes arbitrary steady solutions in terms of a suitably constrained hamiltonian. Two Arnol'd-like stability theorems are obtained which can establish the linear stability in the sense of Liapunov of these steady solutions. Based on this analyses two *a priori* estimates are derived which bound the disturbance energy and the Liapunov norm with respect to the initial disturbance potential enstrophy and energy. In the limit of parallel shear flow solutions corresponding to a current flowing along isobaths, the first formal stability theorem reduces to a previously established normal-mode stability result. Based on the formal stability analysis, convexity conditions are given for the constrained hamiltonian that can rigorously establish nonlinear stability in the sense of Liapunov for the steady current solutions. A variational principle is also presented which can describe steadily-travelling isolated cold eddy solutions of the model. The principle is based on constraining the hamiltonian with appropriately chosen Casimir and momentum invariants. It is shown that a suitably extended form of Andrews' theorem holds for our model equations. Therefore, the stability of the steadily-travelling isolated eddy solutions cannot be established using the energy-Casimir analysis developed here.

1. Introduction

Density or temperature fronts and isolated eddies are common features in many coastal regions of the world oceans. Examples of the currents associated with density fronts include the Denmark Strait overflow, Antarctic bottom water formed in the Weddell Sea (Whitehead & Worthington 1982), deep water formation in the Adriatic Sea (Zoccolotti & Salusti 1987), and deep water replacement in the Strait of Georgia (LeBlond *et al.* 1991), among many others. The instability of these currents may lead to the formation of deep cold coherent eddies (Armi & D'Asaro 1980; Houghton *et al.* 1982; Nof 1983; Mory *et al.* 1987; among others). These flows play an important role in the mesoscale physical and biological dynamics of the benthic boundary on continental shelves (see, for example, Cooper 1955; Johnson & Schneider 1969).

There are two important kinematic aspects that must be accounted for in a complete dynamical description of these flows. The first of these centres on the fact that the isopycnal deflections associated with density currents and solitary cold eddies are not small in comparison with the scale height of the hydrostatic

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geopotential. This is an important fact because it implies that the space and time derivatives of the density field in the mass continuity equation cannot be neglected in comparison with the horizontal divergence of the velocity field. This implies, at least in principle, a quasi-geostrophic model will be inappropriate as a dynamical theory. A second characteristic that is important is the fact that these flow are strongly baroclinic. This fact is expressed theoretically through the *Stern integral constraint* (Mory 1983, 1985) which proves that for an isolated cold eddy (steadily-travelling on a sloping bottom) to exist, there must be a finite compensating cyclone in the overlying fluid. Experiments have consistently shown (Mory *et al.* 1987; Whitehead *et al.* 1990) that the eulerian velocity field in the overlying surrounding fluid is at least comparable (if not larger than) to the co-moving velocities in the eddy interior.

To construct a theory that could address these issues I, among others (see, for example, Whitehead *et al.* 1990; Swaters & Flierl 1991; Swaters 1991), have developed a model (see (2.10) and (2.11)) that focuses directly on the baroclinic and finite-amplitude isopycnal deflection aspects of the physics. These model equations correspond to a strongly baroclinic, intermediate length-scale geostrophic dynamical balance. This balance represents a middle dynamical régime between a more complete ageostrophic balance and the lower frequency/wavenumber quasi-geostrophic balance (see also the discussion in Cushman-Roisin 1986). Swaters & Flierl (1991) and Swaters (1991) have shown how the model can be obtained from a formal asymptotic reduction of the relevant two-layer shallow-water equations assuming a small (appropriately scaled) bottom slope parameter.

Swaters & Flierl (1991) showed that the model possessed steadily-travelling isolated cold eddy solutions. These coherent eddy solutions are of oceanographic interest because in many coastal regions isolated flow features of this type are commonly observed. Swaters & Flierl used the model to describe the slow diabatic warming (an inherently baroclinic process) of an initially isolated cold eddy and explicitly calculated the trailing topographic wave field that is produced in an attempt to explain the evolution of the cold dome observed on the New England Bight by Houghton (1982).

Swaters (1991) showed that the model also possessed steady along-shore coupled front or mesoscale gravity current solutions. Solutions of this kind were important to find because it had been suggested by previous authors (see, for example, Nof 1983; Mory *et al.* 1987) that isolated cold eddies might be formed by the baroclinic instability of mesoscale gravity currents. It is therefore not unreasonable to speculate that this new model might be able to describe the dynamical evolution of an unstable coupled front to a configuration of isolated steadily-travelling cold eddies. Swaters (1991) gave a detailed linear stability analysis for the mesoscale gravity current solutions of the model.

The principal purpose of this paper is to show that these new model equations can be written as a non-canonical hamiltonian system (in the sense described by, for example, Olver (1982) and Benjamin (1984)) and to use this formalism to study some of the linear and nonlinear dynamical aspects of the steady flow and isolated eddy solutions of the model. Infinite-dimensional hamiltonian theory is at the centre of the mathematical formalism for a theory of solitons and more generally coherent structures (such an isolated eddies). In particular, the stability and perturbation theory developed for solitons (see, for example, Benjamin 1972; Bona 1975; Kaup *et al.* 1978) revolves around exploiting the hamiltonian structure of the governing equations. In addition, hamiltonian theory is at the centre of the *energy-Casimir* formalism for establishing the nonlinear stability in the sense of Liapunov of steady flow solutions to various fluid and plasma dynamics models (see, for example, Arnol'd 1965, 1969; Holm *et al.* 1985; Shepherd 1990).

The plan of this paper is as follows. In §2 we give a brief introduction to the scaling assumptions and asymptotic expansion used to derive the model equations. We then turn to developing the hamiltonian structure for the model and show how to write the dynamics using a Poisson bracket formalism. The general family of Casimir functionals (these are a special class of invariants that span the kernel of the Poisson bracket) and appropriate impulse or momentum functional are established that are required in order to provide a variational principle for arbitrary steady flows and steadily-translating isolated eddies.

In §3 we establish the variational principle for steady flows and use this principle to establish two *formal* stability theorems which describe conditions that can establish the linear stability in the sense of Liapunov for these flows. In particular, we show that the stability conditions established by Swaters (1991) for parallel shear flow solutions of this model can be obtained as the appropriate special limit of these more general theorems. Finally, in §3 we generalize the formal stability theorems to establish sufficient convexity hypotheses on the Casimir functionals that will allow us to give two theorems that establish the nonlinear stability in the sense of Liapunov of steady flow solutions to the model.

In §4 we present two variational principles that describe steady-travelling isolated eddy solutions to the model. In §4, we also give theorems which show that there can be no non-trivial eddy solutions which can satisfy appropriately modified-formal stability theorems of the sort established in §3. Our theorems correspond to an application of Andrews' Theorem (Andrews 1984) appropriately modified to take account of the baroclinic structure inherent in our model. This point underscores an important limitation in the energy-Casimir method. The energy-Casimir method has not generally been able to prove the stability of steadily-translating eddy solutions. For example, the stability proofs of Benjamin (1972) and Bona (1975) use an energy-momentum argument to establish the fact that the Kortweg–de-Vries (KdV)-soliton corresponds to a minimum of a suitably constrained energy hypersurface. The paper is summarized in §5.

2. Problem formulation and hamiltonian structure

(*a*) *Derivation of the governing equations*

Since a detailed discussion has already been given on the physical arguments in relation to the asymptotic expansion required to obtain the model (Swaters & Flierl 1991; Swaters 1991), here we will only very briefly introduce the scalings and model derivation. Under a rigid-lid approximation, the *non-dimensional* two-layer shallow-water equations for the upper-layer (1) and lower layer (2) (see figure 1) can be written in the form, respectively,

$$s(\partial_t + \boldsymbol{u}_1 \cdot \nabla)\,\boldsymbol{u}_1 + \hat{e}_3 \times \boldsymbol{u}_1 + \nabla\eta = \boldsymbol{0}, \tag{2.1}$$

$$sh_t + \nabla\cdot[\boldsymbol{u}_1(sh - sy - 1)] = 0, \tag{2.2}$$

$$s(\partial_t + \boldsymbol{u}_2 \cdot \nabla)\,\boldsymbol{u}_2 + \hat{e}_3 \times \boldsymbol{u}_2 + \nabla p = \boldsymbol{0}, \tag{2.3}$$

$$h_t + \nabla\cdot[h\boldsymbol{u}_2] = 0, \tag{2.4}$$

$$p + y = h + \eta, \tag{2.5}$$

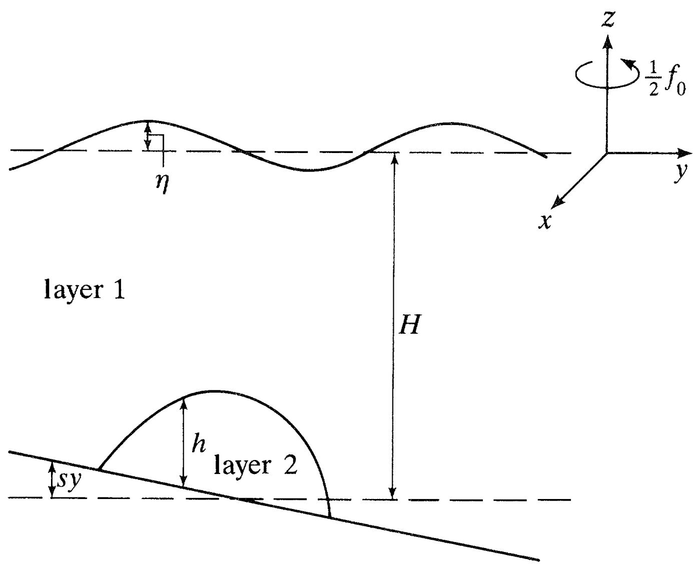

Figure 1. General geometry of the two layer model considered in this paper.

where subscripts with respect to (x, y, t) indicate partial differentiation and $\nabla = (\partial_x, \partial_y)$ with (x, y) the horizontal coordinates and t is time, and where $\boldsymbol{u}_1 = (u_1, v_1)$ and $\boldsymbol{u}_2 = (u_2, v_2)$ are the upper and lower layer velocity fields, respectively, and where η, h, s and p are the reduced layer-1 pressure, frontal or eddy thickness and bottom slope parameter and dynamic pressure in layer-2, respectively. The only non-dimensional parameter in these equations is the slope parameter and it is defined through $s \equiv s^* L/H$ where s^*, L and H are the unscaled or actual bottom slope, horizontal length-scale and reference depth respectively. The horizontal length-scale $L \equiv (g'H)^{\frac{1}{2}}/f_0$ where $g' \equiv g(\rho_2 - \rho_1)/\rho_2 > 0$ is the (stable) reduced gravity with g the gravitational acceleration and ρ_1 and ρ_2 the layer-1 and layer-2 densities, respectively, and f_0 the constant Coriolis parameter. The non-dimensional variables are related to the *dimensional* (asterisked) variables through the relations

$$\left.\begin{aligned} (x^*, y^*) = L(x, y), \quad t^* = f_0 L (g' s)^{-1} t, \quad h^* = sHh, \\ \boldsymbol{u}_1^* = s f_0 L \boldsymbol{u}_1, \quad \eta^* = s(f_0 L)^2 g^{-1} \eta, \quad \boldsymbol{u}_2^* = g' s^* f_0^{-1} \boldsymbol{u}_2, \\ p^* = \rho_2 L g' s^* p. \end{aligned}\right\} \tag{2.6}$$

In addition to the governing equations appropriate boundary conditions are required. If we denote the projection on the plane $z = 0$ of the curve(s) where the layer-2 height vanishes as $\phi(x, y, t) = 0$, then the kinematic boundary condition is

$$\phi_t + \boldsymbol{u}_2 \cdot \nabla \phi = 0 \quad \text{on} \quad \phi(x, y, t) = 0, \tag{2.7}$$

and the layer-2 height must satisfy

$$h(x, y, t) = 0 \quad \text{on} \quad \phi(x, y, t) = 0. \tag{2.8}$$

In addition to these conditions, pressure and normal mass flux continuity in layer-1 is required across $\phi(x, y, t) = 0$. If solid boundaries exist, then there will also exist no normal flow conditions on these boundaries.

It can be shown that for typical flows on a continental shelf $s \approx 10^{-2}$ (Swaters & Flierl 1991; Swaters 1991). If a straightforward asymptotic expansion of the form

$$(\eta, p, \boldsymbol{u}_1, \boldsymbol{u}_2, h, \phi) \sim (\eta_0, p_0, \boldsymbol{u}_{10}, \boldsymbol{u}_{20}, h_0, \phi_0) + s(\eta_1, p_1, \boldsymbol{u}_{11}, \boldsymbol{u}_{21}, h_1, \phi_1) + \ldots, \tag{2.9}$$

is inserted into the governing equations (2.1)–(2.5), it is not difficult to show that the leading order dynamics is determined by (after dropping the zero subscript)

$$(\Delta \partial_t - \partial_x)\, \eta - h_x + \partial(\eta, \Delta\eta) = 0, \tag{2.10}$$

$$h_t + h_x + \partial(\eta, h) = 0, \tag{2.11}$$

where the jacobian $\partial(A,B) \equiv A_x B_y - A_y B_x$ and where $\Delta \equiv \nabla^2$. The other leading order fields $(\boldsymbol{u}_{10}, \boldsymbol{u}_{20}, p_0)$ are related to η and h through the expressions

$$\boldsymbol{u}_1 = \hat{e}_3 \times \nabla\eta, \tag{2.12a}$$

$$\boldsymbol{u}_2 = \hat{e}_1 + \hat{e}_3 \times \nabla(\eta + h), \tag{2.12b}$$

$$p = -y + \eta + h, \tag{2.12c}$$

where, here again, for notational convenience, the zero subscript has been deleted. Equation (2.10) (actually (2.10)+(2.11)) simply expresses the conservation of the potential vorticity $\Delta\eta + h - y$ in layer-1 following the geostrophic flow $\hat{e}_3 \times \nabla\eta$, and (2.11) can be interpreted as expressing the conservation of mass for layer-2 for a geostrophically balanced density-driven flow on a sloping bottom.

The correct interpretation of the boundary conditions (2.7) and (2.8) under the asymptotic limit $s \rightarrow 0$ is a little more subtle and has important implications for the hamiltonian formulation of the governing equations. If the asymptotic expansion (2.9) is inserted into (2.7) and (2.8), it follows that

$$h = 0, \quad \text{on} \quad \phi = 0, \tag{2.13a}$$

$$\phi_t + \phi_x + \partial(\eta + h, \phi) = 0, \quad \text{on} \quad \phi = 0. \tag{2.13b}$$

However, assuming that h and η assume their respective values on $\phi = 0$ smoothly, then (2.13a) and (2.11) can be used to infer (2.13b). It follows from (2.13a) assuming sufficient smoothness that

$$\tilde{\nabla} h \times \tilde{\nabla}\phi = \mathbf{0}, \tag{2.14}$$

on $\phi = 0$ and $\tilde{\nabla}$ is the space-time gradient given by $\tilde{\nabla} \equiv (\partial_t, \nabla)$. Thus in particular

$$\partial(h, \phi) \equiv 0, \tag{2.15}$$

on $\phi = 0$. This fact can be used to eliminate the second jacobian term in (2.13b). As well, substitution of (2.15) into (2.11) implies that

$$h_t + h_x + \partial(\eta, \phi)\,(h_x/\phi_x) = 0,$$

on $\phi = 0$ (again assuming the limit can be taken). However, this expression can be rearranged exactly into

$$\phi_t + \phi_x + \partial(\eta, \phi) = 0, \tag{2.16}$$

on $\phi = 0$ if the components of (2.14) are used. Expression (2.16) is simply (2.13b) with (2.15) used. Consequently, there is a degeneracy in the governing equations (see also Swaters 1991) and we may, without loss of generality, take as the basic model the equations (2.10), (2.11) and (2.13a) together with appropriate boundary conditions on η (which we will discuss as the need arises). The above degeneracy is important because it will mean that in fact we are not dealing with a true free-boundary problem and thus many of the technicalities associated with the hamiltonian formulation of these kinds of problems (see, for example, Lewis *et al.* 1985) can be avoided.

The model equations (2.10), (2.11) and (2.13a) have many solutions of oceanographic interest. Swaters & Flierl (1991) showed the existence of a class of steadily-travelling eddy solutions with compact support and studied the evolution of these solutions when the effects of diabatic heating are retained in the model derivation. Swaters (1991) examined the *linear* stability problem associated with coupled front solutions of the form $h = h_0(y)$, $\eta = \eta_0(y)$ and $\phi = y - \phi_0$ (where ϕ_0 is constant). Our aim here is to first construct a general hamiltonian framework for (2.10) and (2.11) and then to exploit this formalism to establish variational principles for general classes of isolated eddy solutions and steady-state frontal solutions and to establish appropriate sufficient conditions for the nonlinear stability of the frontal solutions.

(*b*) *Hamiltonian structure and invariants of the motion*

Following Benjamin (1984) or Olver (1982) a system of partial differential equations is said to be hamiltonian if a (conserved) hamiltonian functional $H(\boldsymbol{q})$ where $\boldsymbol{q} = (q_1, \ldots, q_n)^{\mathrm{T}}$ is a (column) vector of n dependent variables and a skew symmetric matrix J of (pseudo-) differential operators satisfying the Jacobi identity can be found so that the dynamical system can be written in the form

$$\boldsymbol{q}_t = J\,\delta H/\delta \boldsymbol{q}, \tag{2.17}$$

where $\delta H/\delta \boldsymbol{q}$ is the variational derivative of H with respect to $\boldsymbol{q}$. The skew symmetry property for the operator J is described by

$$\langle \boldsymbol{a}, J\boldsymbol{b}\rangle = -\langle J\boldsymbol{a}, \boldsymbol{b}\rangle, \tag{2.18}$$

where $\langle \boldsymbol{a}, \boldsymbol{b}\rangle$ is (typically) the L^2-inner product

$$\langle \boldsymbol{a}, \boldsymbol{b}\rangle = \int_\Omega \boldsymbol{a}\boldsymbol{b}^{\mathrm{T}}\,\mathrm{d}x\,\mathrm{d}y, \tag{2.19}$$

where $\Omega \subseteq \mathbb{R}^2$ is the appropriate two-dimensional *spatial* domain on which the flow occurs.

Theorem 2.1. *Equations* (2.10) *and* (2.11) *is hamiltonian for the choice of*

$$H(\boldsymbol{q}) \equiv \frac{1}{2}\iint_\Omega \nabla\eta\cdot\nabla\eta + \chi_h[(h-y)^2 - y^2]\,\mathrm{d}x\,\mathrm{d}y, \tag{2.20}$$

and $J = [J_{ij}]$ *a* 2×2-*matrix whose components are given by*

$$J_{ij} = -\delta_{i1}\,\delta_{j1}\,\partial(q_1 - y, \cdot) + \delta_{i2}\,\delta_{j2}\,\partial(q_2, \cdot), \tag{2.21}$$

where δ_{nm} *is the Kronecker delta function and* $\boldsymbol{q} = (q_1, q_2)^{\mathrm{T}}$ *with*

$$q_1 \equiv \Delta\eta + h, \tag{2.22}$$

$$q_2 \equiv h. \tag{2.23}$$

Before proceeding to prove this theorem the following remarks should be made. It will generally be the case that the support of h is a subset of the support of η. For example, in Swaters & Flierl (1991) $\eta(x,y,t)$ extended out to infinity but $h(x,y,t)$ had compact support. In the stability study of Swaters (1991) $h(x,y,t)$ extended to infinity in the x-direction, but h was non-zero only in finite region in the y-direction. However, the geometry of these two situations need not necessarily occur and it is possible that $h(x,y,t)$ extends to infinity. In this situation the $-y^2$ factor in the second term of the integrand of (2.20) is needed to ensure that H is finite for the situation of interest where $|h| \to 0$ as $x^2+y^2 \to \infty$ for all $t \geqslant 0$. In the case where $h(x,y,t)$ does not extend to infinity in some or all directions, then the $-y^2$ factor will ensure that $\{[h(x,\phi(x,t),t)-y]^2-y^2\} \equiv 0$ on the boundary curve(s) $y=\phi(x,t)$. The function χ_h is the characteristic function associated with the support of h, i.e.

$$\chi_h = 1, \quad \text{if} \quad h \neq 0, \tag{2.24a}$$

$$\chi_h = 0, \quad \text{if} \quad h = 0. \tag{2.24b}$$

This term is formally included because technically the function $h(x,y,t)$ in (2.20) is not defined for (x,y) values outside its support. However, for our purposes, we may extend the domain of h to all of Ω provided we define $h \equiv 0$ for all those (x,y)-coordinates outside it support. From time to time we will drop the characteristic function notation when the meaning is clear.

Another important point to make is that if Ω is not simply connected, then the definition of $H(\boldsymbol{q})$ must be modified to include the sum of all the (conserved) circulation integrals on each individual smoothly connected closed boundary curves the union of which define $\partial\Omega$ with a Lagrange multiplier given by the *negative* of the constant value that η must assume on that portion of the boundary (see Holm *et al.* 1985) in order that the required variational derivatives are correctly computed. (The fact that η must be constant on each simply-connected portion of $\partial\Omega$ follows from the no normal flow constraint across $\partial\Omega$, i.e. $\boldsymbol{n}\cdot(\hat{e}_3 \times \nabla\eta) = 0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the unit outward normal on $\partial\Omega$.) Alternatively, it is possible to extend the hamiltonian formalism by introducing additional dependent variables corresponding to the circulations and modify the J-operator accordingly (see Shepherd 1990).

Proof of Theorem 2.1. Our first task is to show that $H(\boldsymbol{q})$ is a conserved functional for the dynamics. It follows from (2.20) that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \iint_\Omega -\eta\,\Delta\eta_t + (h-y)\,h_t\,\mathrm{d}x\,\mathrm{d}y,$$

where the superfluous characteristic function is no longer needed, and where $\eta(\partial\Omega) = 0$ has been used. There is no contribution from the time derivative associated with the boundary curve(s) $y=\phi(x,t)$ if h does not extend to infinity since $[(h-y)^2-y^2]|_{y=\phi} \equiv 0$. Substitution of (2.10) and (2.11) into the above expression leads to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \iint_\Omega \eta[\partial(\eta,\Delta\eta)-\eta_x-h_x]-(h-y)\,[h_x+\partial(\eta,h)]\,\mathrm{d}x\,\mathrm{d}y,$$

which can be integrated by parts, exploiting the fact that $h(x,\phi,t) \equiv 0$ and that we are assuming $\eta(\partial\Omega) = 0$, to give

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \iint_\Omega -\eta h_x - \eta_x h\,\mathrm{d}x\,\mathrm{d}y = -\iint_\Omega (\eta h)_x\,\mathrm{d}x\,\mathrm{d}y \equiv 0,$$

which establishes that $H(\boldsymbol{q})$ is an invariant of the motion. Further, it is straightforward to verify that (2.21) satisfies (2.18) on account of the algebraic properties of $\partial(*,\cdot)$. As well, it is a straightforward but lengthy computation to show that the operator J satisfies the Jacobi identity using arguments similar to those presented in McIntyre & Shepherd (1987, Appendix C) or Scinocca & Shepherd (1992, Appendix A).

We now turn to the direct verification of (2.17). First we need to compute the variational derivatives of $H(\boldsymbol{q})$. From (2.20) we have

$$\delta H = \iint_{\Omega} \nabla\eta\cdot\nabla(\delta\eta) + (h-y)\,\delta h \,\mathrm{d}x\,\mathrm{d}y$$
$$= \iint_{\Omega} -\eta\,\Delta\,\delta\eta + (h-y)\,\delta h + \eta\,\delta h - \eta\,\delta h \,\mathrm{d}x\,\mathrm{d}y,$$

assuming that $\eta \equiv 0$ on $\partial\Omega$ and dropping the superfluous characteristic function. Note that there is no contribution associated with variations of the boundary curve(s) $y = \phi(x,t)$ on which $h = 0$ (if h does not extend to infinity) since $[h-y)^2 - y^2]|_{y=\phi} \equiv 0$. The above expression can be arranged into

$$\delta H = \iint -\eta\delta q_1 + (h-y+\eta)\,\delta q_2 \,\mathrm{d}x\,\mathrm{d}y,$$

so that the variational derivatives are given by

$$\delta H/\delta q_1 = -\eta, \tag{2.25a}$$
$$\delta H/\delta q_2 = h+\eta-y. \tag{2.25b}$$

Substitution of (2.25) into the right-hand side of (2.17) yields

$$J\frac{\delta H}{\delta \boldsymbol{q}} = [\partial(q_1-y,\eta), \partial(q_2, \eta+h-y)]^{\mathrm{T}},$$

which if (2.22) and (2.23) are substituted in implies

$$(\Delta\eta+h)_t = \eta_x - \partial(\eta, \Delta\eta+h), \tag{2.26a}$$
$$h_t = -\partial(\eta,h) - h_x. \tag{2.26b}$$

Equation (2.26b) is exactly (2.11), and (2.10) corresponds to (2.26a) minus (2.26b). □

(c) *Poisson bracket formulation*

The dynamics can alternatively be described using a Poisson bracket (Olver 1982; Benjamin 1984) in the form

$$\mathrm{d}F/\mathrm{d}t = [F,H], \tag{2.27}$$

where the Poisson bracket is defined by

$$[F,G] \equiv \left\langle \frac{\delta F}{\delta \boldsymbol{q}},\quad J\frac{\delta G}{\delta \boldsymbol{q}} \right\rangle, \tag{2.28}$$

where F and G are arbitrary functionals of $\boldsymbol{q}$. Substitution of J given by (2.21) into (2.28) implies that

$$[F,G] \equiv -\iint_{\Omega} \frac{\delta F}{\delta q_1}\partial\left(q_1 - y,\quad \frac{\delta G}{\delta q_1}\right)\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega} \frac{\delta F}{\delta q_2}\partial\left(q_2,\quad \frac{\delta G}{\delta q_2}\right)\mathrm{d}x\,\mathrm{d}y. \tag{2.29}$$

It is straightforward to verify that

$$q_{1_t} = [q_1, H], \tag{2.30a}$$
$$q_{2_t} = [q_2, H]. \tag{2.30b}$$

As mentioned previously it can be shown that the Poisson bracket (2.29) satisfies the Jacobi identity via arguments similar to those presented in McIntyre & Shepherd (1987, Appendix C) or Scinocca & Shepherd (1992, Appendix A).

It is important to make one last comment about our presentation of the hamiltonian structure. The matrix operator J given by (2.21) or equivalently the non-canonical eulerian Poisson bracket (2.28) could easily be guessed since the underlying model equations (2.10) and (2.11) very much resemble multi-layer quasi-geostrophic flow for which the (Poisson) bracket is well known (see Holm *et al.* 1985, §4). However, while this intuitive approach works well enough for the model discussed here a much better procedure, particularly from the viewpoint of more difficult problems, is to obtain the non-canonical eulerian bracket by systematic reduction of the canonical lagrangian bracket (see, for example, Marsden & Weinstein 1983; Simo *et al.* 1988; Shepherd 1990, Appendix A).

(*d*) *Some invariants of the dynamics*

To construct the requisite variational principles for the steady flows we first need to obtain some invariants associated with the dynamics. The two classes of invariants needed belong to either the group of conserved quantities associated with invariance properties of the hamiltonian itself (i.e. Noether's Theorem, e.g. time translation invariance implies energy conservation and so on), or the group of conserved quantities that lie in the kernel of the Poisson bracket called *Casimirs* (see Holm *et al.* 1985). Because the operator J is non-invertible (this is what is meant by non-canonical hamiltonian dynamics), the kernel of the Poisson bracket is non-trivial.

For our purposes all we need from Noether's Theorem is the invariant denoted M, associated with the x-translation invariance of the hamiltonian, determined by

$$J\frac{\delta M}{\delta \boldsymbol{q}} = -\boldsymbol{q}_x \tag{2.31}$$

(see Benjamin 1984), which for our system is found to be

$$M = \iint_{\Omega} y\Delta\eta \,\mathrm{d}x\,\mathrm{d}y. \tag{2.32}$$

To verify (2.31), note that

$$\delta M = \iint_{\Omega} y\Delta(\delta\eta) + y\delta h - y\delta h \,\mathrm{d}x\,\mathrm{d}y,$$

so that it follows $\qquad \delta M/\delta q_1 = y, \quad \delta M/\delta q_2 = -y,$

from which one can easily see that (2.31) holds. Physically, M is simply the x-direction linear momentum in layer-2 (Benjamin (1984) uses the terminology *impulse*).

The Casimirs are those functionals $C(\boldsymbol{q})$ satisfying

$$[F, C] = 0, \tag{2.33}$$

for every sufficiently smooth functional $F = F(\boldsymbol{q})$. Using (2.28) it follows that the Casimirs must satisfy

$$J\frac{\delta C}{\delta \boldsymbol{q}} \equiv \boldsymbol{0},$$

which in component form is given by

$$\partial\,(q_1 - y, \quad \delta C/\delta q_1) = 0, \tag{2.34}$$
$$\partial\,(q_2, \quad \delta C/\delta q_2) = 0, \tag{2.35}$$

which can be easily solved to give the general solution

$$C(\boldsymbol{q}) = \iint_{\Omega} \Phi_1(q_1-y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega} \Phi_2(q_2)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.36}$$

where Φ_1 and Φ_2 are arbitrary functions (some smoothness will be eventually required) of their arguments.

3. Variational principle for and the stability of steady solutions

(a) *Variational principle for arbitrary steady solutions*

General steady solutions $\eta = \eta_0(x,y)$ and $h = h_0(x,y)$ of (2.10) and (2.11) satisfy

$$\partial(\eta_0, \Delta\eta_0 + h_0 - y) = 0, \tag{3.1}$$

$$\partial(\eta_0 - y, h_0) = 0, \tag{3.2}$$

which can be integrated formally to imply

$$\eta_0 = F_1(\Delta\eta_0 + h_0 - y), \tag{3.3}$$

$$\eta_0 = y + F_2(h_0), \tag{3.4}$$

where F_1 and F_2 are appropriate functions of their arguments which implicitly define the steady solutions. Consider the (conserved) constrained hamiltonian

$$\mathscr{H} = H + C, \tag{3.5}$$

where H is the hamiltonian (2.20) and C is the Casimir (2.36). It follows from (3.5) that

$$\delta\mathscr{H} = \iint_{\Omega} \{[\Phi_1' - \eta]\,\delta q_1 + [\Phi_2' + \eta - y + h]\,\delta q_2\}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.6}$$

where $\Phi_1' \equiv \mathrm{d}\Phi_1/\mathrm{d}(q_1 - y)$ and $\Phi_2' \equiv \mathrm{d}\Phi_2/\mathrm{d}q_2$. We can therefore establish the variational principle.

Proposition 3.1. *General steady solutions* $\eta_0(x,y)$ *and* $h_0(x,y)$ *as determined by* (3.3) *and* (3.4) *satisfy the first-order necessary condition* $\delta\mathscr{H}(\eta_0, h_0) = 0$ *for extremizing the constrained hamiltonian* $\mathscr{H} = H + C$ *provided the Casimir densities are given by*

$$\Phi_1(q_1 - y) = \int_{-y}^{q_1 - y} F_1(\xi)\,\mathrm{d}\xi, \tag{3.7}$$

$$\Phi_2(q_2) = -\int_0^{q_2} F_2(\xi)\,\mathrm{d}\xi - \tfrac{1}{2}(q_2)^2. \tag{3.8}$$

(b) *Formal stability theorems*

The *linear* stability in the sense of Liapunov for the steady solutions $\eta_0(x,y)$ and $h_0(x,y)$ determined formally through (3.3) and (3.4) can be proved if the functions F_1 and F_2 satisfy conditions which will ensure that $\delta^2\mathscr{H}(\eta_0, h_0)$ is definite for all suitably smooth perturbations $\delta\eta$ and δh. Holm *et al.* (1985) have termed this sense of stability as *formal stability*. The second variation of $\mathscr{H}$ is given by

$$\delta^2\mathscr{H}(\eta, h) = \iint_{\Omega} \{\nabla(\delta\eta)\cdot\nabla(\delta\eta) + \Phi_1''(\Delta\delta\eta + \delta h)^2 + (\Phi_2'' + 1)\,(\delta h)^2\}\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \iint_{\Omega} \{[\Phi_1' - \eta]\,(\Delta\delta^2\eta + \delta^2 h) + [\Phi_2' + \eta - y + h]\,\delta^2 h\}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.9}$$

from which it follows that

$$\delta^2\mathscr{H}(\eta_0,h_0)=\iint_\Omega\{\nabla(\delta\eta)\cdot\nabla(\delta\eta)+\Phi''_{10}(\Delta\delta\eta+\delta h)^2+(\Phi''_{20}+1)(\delta h)^2\}\,\mathrm{d}x\,\mathrm{d}y, \quad (3.10)$$

where it is understood that $\Phi''_{10}\equiv \mathrm{d}^2\Phi_1/\mathrm{d}(q_1-y)^2$ evaluated for $q_1-y=q_{10}\equiv \Delta\eta_0+h_0-y$ and $\Phi''_{20}\equiv \mathrm{d}^2\Phi_2/\mathrm{d}q_2^2$ evaluated for $q_2=q_{20}\equiv h_0$.

It is straightforward to verify that $\delta^2\mathscr{H}(\eta_0,h_0)$ is an invariant of the linear dynamics obtained by substituting $h(x,y,t)=h_0(x,y)+\delta h(x,y,t)$ and $\eta(x,y,t)=\eta_0(x,y)+\delta\eta(x,y,t)$ into (2.10) and (2.11) neglecting all quadratic perturbation terms (i.e. the linear stability problem), which can be written in the form

$$(\Delta\delta\eta+\delta h)_t+\partial(\delta\eta-\Phi''_{10}[\Delta\delta\eta+\delta h],\Delta\eta_0+h_0-y)=0, \quad (3.11)$$

$$(\delta h)_t+\partial(\delta\eta+[\Phi''_{20}+1]\,\delta h,h_0)=0. \quad (3.12)$$

It is possible to give two results that establish the definiteness of $\delta^2\mathscr{H}(\eta_0,h_0)$. The first of these can be considered the analogue of Arnol'ds (1965) *first* stability theorem and the second result as the analogue of his *second* stability theorem.

Theorem 3.2. *The steady solutions $h=h_0(x,y)$ and $\eta=\eta_0(x,y)$ are linearly stable in the sense of Liapunov with respect to the perturbation norm $\|\delta \boldsymbol{q}\|=[\delta^2\mathscr{H}(\eta_0,h_0)]^{\frac{1}{2}}$ if the Casimir functions given by* (3.7) *and* (3.8) *where F_1 and F_2 are given by* (3.3) *and* (3.4) *satisfy*

$$\Phi''_{10}\geqslant 0, \quad (3.13)$$

$$\Phi''_{20}+1\geqslant 0, \quad (3.14)$$

for all $(x,y)\in\Omega$.

In fact, it is more instructive from the point of view of applications to recast (3.13) and (3.14) into a slightly different form. It follows from (3.7) and (3.8) that $\Phi''_{10}=F'_{10}$ and $\Phi''_{20}=-F'_{20}-1$. However, from (3.3) and (3.4) we have

$$F'_{10}\equiv U_0(x,y)/(\Delta U_0+1-h_{0y}), \quad (3.15)$$

$$F'_{20}\equiv -[U_0(x,y)+1]/h_{0y}, \quad (3.16)$$

where $U_0(x,y)\equiv -\partial\eta_0/\partial y$ is the x-direction velocity in layer one. Substitution of (3.15) and (3.16) into (3.13) and (3.14) gives an alternate form for the stability conditions as

$$U_0/(\Delta U_0+1-h_{0y})\geqslant 0, \quad (3.17)$$

$$(U_0+1)/h_{0y}\geqslant 0. \quad (3.18)$$

This form is more useful from an oceanographic point of view because in practice oceanic coastal flows tend to follow isobaths which in our model correspond to steady solutions of the form $\eta_0\equiv\eta_0(y)$ and $h_0=h_0(y)$ so that $U_0\equiv U_0(y)$.

The stability conditions (3.17) and (3.18) can be interpreted as the analogue of Fjørtoft's theorem in a suitable reference frame for our flow configuration (see Drazin & Reid 1981, §22). Swaters (1991) examined the linear stability of the *along-shelf* currents and derived the result that the flow was linear stable in the sense of Liapunov if $h_{0y}>0$ for $U_0=$ const. (which by galilean invariance of (2.10) and (2.11) means that we may set $U_0\equiv 0$ with no loss of generality). Clearly, (3.17) and (3.18) reduce to this result of Swaters (1991).

Another important result that follows from Theorem 3.2 is that we can obtain an *a priori* estimate for the perturbation or disturbance energy given by

$$E(\delta \boldsymbol{q}) \equiv \left\{ \iint_{\Omega} \nabla(\delta\eta)\cdot\nabla(\delta\eta) + (\delta h)^2\,\mathrm{d}x\,\mathrm{d}y \right\}^{\frac{1}{2}}. \tag{3.19}$$

Note that $E(\delta \boldsymbol{q})$ is not in general an invariant of the linearized dynamics (3.11) and (3.12). Using the inequality

$$\iint_{\Omega} \nabla(\delta\eta)\cdot\nabla(\delta\eta) + (\Phi''_{20}+1)\,(\delta h)^2\,\mathrm{d}x\,\mathrm{d}y \geqslant \Gamma[E(\delta \boldsymbol{q})]^2,$$

where $\Gamma \equiv \min\,[1, \inf_{\Omega}(\Phi''_{20}+1)] > 0$ assuming the conditions of Theorem 3.2 hold (with a strict inequality assumed in (3.14) if necessary), implies that

$$\begin{aligned}[E(\delta \boldsymbol{q})]^2 &\leqslant \Gamma^{-1} \iint_{\Omega} \nabla(\delta\eta)\cdot\nabla(\delta\eta) + (\Phi''_{20}+1)\,(\delta h)^2\,\mathrm{d}x\,\mathrm{d}y, \\ &\leqslant \Gamma^{-1}\delta^2\mathscr{H}(\eta_0, h_0),\end{aligned}$$

assuming (3.13) holds, which in turn implies the *a priori* energy estimate

$$E(\delta \boldsymbol{q}) \leqslant [\Gamma^{-1}\delta^2\mathscr{H}(\eta_0, h_0)]^{\frac{1}{2}} \equiv [\Gamma^{-1}\delta^2\mathscr{H}(\eta_0, h_0)|_{t=0}]^{\frac{1}{2}}. \tag{3.20}$$

Consequently, we could have used the more physically related perturbation energy as the disturbance norm in Theorem 3.2 which of course has greater physical significance.

The second stability result we can obtain amounts to a demonstration of the conditions required for $\delta^2\mathscr{H}(\eta_0, h_0)$ to be negative definite. We proceed as follows. Assuming that the domain Ω is bounded in at least one direction (e.g. a channel domain), the Poincaré inequality

$$\iint_{\Omega} \nabla(\delta\eta)\cdot\nabla(\delta\eta)\,\mathrm{d}x\,\mathrm{d}y \leqslant \tilde{C} \iint_{\Omega} [\Delta(\delta\eta)]^2\,\mathrm{d}x\,\mathrm{d}y, \tag{3.21}$$

exists for some positive finite constant $\tilde{C}$ (Ladyzhenskaya 1969; e.g. if Ω is bounded then $1/\tilde{C}$ is the minimum positive eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$ the boundary of Ω). Substitution of (3.21) into (3.10) implies

$$\delta^2\mathscr{H}(\eta_0, h_0) \leqslant \iint_{\Omega} \{\tilde{C}[\Delta(\delta\eta)]^2 + \Phi''_{10}[\Delta\delta\eta + \delta h]^2 + (\Phi''_{20}+1)\,(\delta h)^2\}\,\mathrm{d}x\,\mathrm{d}y,$$

which can be rearranged using the identity

$$\begin{aligned}&\tilde{C}[\Delta(\delta\eta)]^2 + \Phi''_{10}[\Delta(\delta\eta) + \delta h]^2 + (\Phi''_{20}+1)\,(\delta h)^2 \\ &\qquad \equiv (\tilde{C} + \Phi''_{10})\,[\Delta(\delta\eta) + \gamma\delta h]^2 + [\Phi''_{20} + 1 + \tilde{C}\gamma]\,(\delta h)^2,\end{aligned} \tag{3.22a}$$

where $\gamma \equiv \gamma(x, y)$ is given by

$$\gamma = \Phi''_{10}/(\tilde{C} + \Phi''_{10}), \tag{3.22b}$$

into the form

$$\delta^2\mathscr{H}(\eta_0, h_0) \leqslant \iint_{\Omega} \{(\tilde{C} + \Phi''_{10})\,[\Delta\delta\eta + \gamma\delta h]^2 + [\Phi''_{20} + 1 + \tilde{C}\gamma]\,(\delta h)^2\}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.23}$$

We therefore have the following result.

Theorem 3.3. *The steady solutions $h = h_0(x,y)$ and $\eta = \eta_0(x,y)$ are linearly stable in the sense of Liapunov with respect to the disturbance norm $\|\delta \boldsymbol{q}\| = [-\delta^2 \mathscr{H}(\eta_0, h_0)]^{\frac{1}{2}}$ if the Casimir functions given by* (3.7) *and* (3.8) *where F_1 and F_2 are given by* (3.3) *and* (3.4) *satisfy*

$$\sup_{\Omega} \Phi''_{10} \leqslant -\tilde{C} < 0, \tag{3.24}$$

$$\Phi''_{20} + 1 \leqslant -\tilde{C}\gamma, \tag{3.25}$$

with $\gamma(x,y)$ given by (3.22*b*) *and $\tilde{C} > 0$ is given by* (3.21), *and where* (3.25) *holds at each point $(x,y) \in \Omega$ and where a strict inequality holds in at least one of* (3.24) *or* (3.25) *so that $\delta^2 \mathscr{H}(\eta_0, h_0)$ is bounded away from zero.*

From the viewpoint of oceanographic applications it is more useful here again to recast the stability conditions (3.24) and (3.25) directly in terms of $U_0(x,y) \equiv \partial \eta_0 / \partial y$ and $h_0(x,y)$. Substitution of (3.15) and (3.16) into (3.24) and (3.25) leads to the alternate representation

$$U_0/(\Delta U_0 + 1 - h_{0y}) \leqslant -\tilde{C}, \tag{3.26}$$

$$(U_0 + 1)/h_{0y} \leqslant -\tilde{C} U_0 / [\tilde{C}(\Delta U_0 + 1 - h_{0y}) + U_0]. \tag{3.27}$$

Another point that is clear from (3.24) or (3.26) is the necessary existence of nonzero flow in the upper layer one for this theorem to hold since if $\Phi''_{10} \equiv 0$ (so that $U_0 \equiv 0$), then (3.24) or (3.26) can obviously never be satisfied.

We can use Theorem 3.3 to obtain an *a priori* estimate for the disturbance norm in terms of the initial disturbance relative enstrophy and potential energy. Assuming that the hypotheses of Theorem 3.3 hold, then

$$\begin{aligned} \|\delta \boldsymbol{q}\|^2 &\equiv -\delta^2 \mathscr{H}(\eta_0, h_0) \equiv -\delta^2 \mathscr{H}(\eta_0, h_0)|_{t=0} \\ &\leqslant -\iint_{\Omega} \Phi''_{10}(\Delta \delta\tilde{\eta} + \delta\tilde{h})^2 + (\Phi''_{20} + 1)(\delta\tilde{h})^2 \,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{3.28}$$

where (3.10) has been used and $\delta\tilde{\eta} \equiv \delta\eta(x,y,t=0)$ and $\delta\tilde{h} \equiv \delta h(x,y,t=0)$. From (3.28) it follows that

$$\|\delta \boldsymbol{q}\|^2 \leqslant \Gamma \iint_{\Omega} (\Delta \delta\tilde{\eta} + \delta\tilde{\eta})^2 + (\delta\tilde{h})^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{3.29}$$

where

$$\Gamma \equiv -\min\left[\inf_{\Omega}(\Phi''_{10}), \inf_{\Omega}(\Phi''_{20} + 1)\right] > 0. \tag{3.30}$$

The integral on the right-hand side of (3.29) corresponds to the sum of the initial relative enstrophy associated with layer one and the initial potential energy in layer two.

(*c*) *Nonlinear stability theorems*

Unlike finite-dimensional hamiltonian dynamics the definiteness of the second variation of a (constrained) hamiltonian does not imply nonlinear stability because of topological difficulties associated with infinite-dimensional function spaces. Specifically, $|\delta^2 \mathscr{H}(\eta_0, h_0)| > 0$ is not sufficient to ensure that $\mathscr{H}(\eta, h)$ is convex in a small but finite neighbourhood of (η_0, h_0) due to the lack of compactness of the unit sphere in Hilbert space (Ebin & Marsden 1970). Consequently, to prove *nonlinear* stability, which is after all what one is really interested in, additional convexity hypotheses are required on $\mathscr{H}(\eta, h)$. This is not a moot point of pure mathematical interest; Ball & Marsden (1984) have shown how for realistic examples in elasticity formal stability does not imply nonlinear stability. Nevertheless, the formal stability theorems obtained in the last section point the correct direction required to establish nonlinear stability in the sense of Liapunov.

The argument begins with constructing the functional

$$\mathscr{L}(\boldsymbol{q}) \equiv H(\boldsymbol{q}+\boldsymbol{q}_0)-H(\boldsymbol{q}_0)+C(\boldsymbol{q}+\boldsymbol{q}_0)-C(\boldsymbol{q}_0), \tag{3.31}$$

where H and C are given by (2.20) and (2.36), respectively, with the Casimir densities Φ_1 and Φ_2 determined by (3.7) and (3.8), respectively, and where $\boldsymbol{q}_0 \equiv (q_{10}, q_{20})^{\mathrm{T}}$ with $q_{10} \equiv \Delta\eta_0 + h_0 - y$ and $q_{20} \equiv h_0$ where $\eta_0 = \eta_0(x,y)$ and $h_0 = h_0(x,y)$ are steady solutions as determined by (3.3) and (3.4). As well, $\boldsymbol{q} = (q_1, q_2)^{\mathrm{T}}$ in (3.31) where $q_1 \equiv \Delta\eta + h$ and $q_2 \equiv h$ where $\eta = \eta(x,y,t)$ and $h = h(x,y,t)$. It is important to point out that $\mathscr{L}(\boldsymbol{q})$ is conserved by the full nonlinear dynamics (2.10) and (2.11) where it is understood that $\boldsymbol{q}_T \equiv \boldsymbol{q}+\boldsymbol{q}_0$ is the dependent variable that solves the governing equations. The variable $\boldsymbol{q}$ in (3.31) represents the departure of the nonlinear time-dependent solution $\boldsymbol{q}_T$ from the steady solution $\boldsymbol{q}_0$ and thus we shall refer to $\boldsymbol{q}$ as the (finite-amplitude) perturbation or disturbance field or flow. Equation (3.31), when written out, can be expressed in the form

$$\mathscr{L}(\boldsymbol{q}) = \frac{1}{2}\iint_\Omega \nabla\eta\cdot\nabla\eta \,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega \left\{\int_{q_{10}}^{q_1+q_{10}} F_1(\xi)\,\mathrm{d}\xi - F_1(q_{10})\,q_1\right\}\mathrm{d}x\,\mathrm{d}y$$
$$-\iint_\Omega \left\{\int_{h_0}^{h+h_0} F_2(\xi)\,\mathrm{d}\xi - F_2(h_0)\,h\right\}\mathrm{d}x\,\mathrm{d}y, \tag{3.32}$$

where (3.3) and (3.4) have been used. Also we used the fact that $\eta_0 \equiv 0$ on the boundary $\partial\Omega$ of Ω. If Ω is not simply-connected then (3.32) still holds but the functional $\mathscr{L}$ must be modified to include a contribution associated with the sum of the circulation integrals associated with each simply-connected boundary curve weighted by the constant value of η_0 on that boundary curve (Holm *et al.* 1985). Another important point to note is that if (3.32) is Taylor expanded about $(\eta, h) = (0,0)$, then the leading order term is simply $\frac{1}{2}\delta^2\mathscr{H}(\eta_0, h_0)$ as given in (3.10).

Suppose that the functions $F_1(\xi)$ and $F_2(\xi)$ satisfy the convexity conditions

$$\alpha_1 < F_1'(\xi) < \beta_1, \tag{3.33a}$$
$$\alpha_2 < F_2'(\xi) < \beta_2, \tag{3.33b}$$

for all arguments ξ where the prime indicates $\mathrm{d}/\mathrm{d}\xi$ and where the constants α_1, β_1, α_2, β_2 are real numbers (appropriate additional hypothesis on these constants will be described momentarily). If (3.33) are integrated twice, it is easy to show that

$$\tfrac{1}{2}\alpha_1(q_1)^2 < \int_{q_{10}}^{q_1+q_{10}} F_1(\xi)\,\mathrm{d}\xi - F_1(q_{10})\,q_1 < \tfrac{1}{2}\beta_1(q_1)^2, \tag{3.34a}$$
$$\tfrac{1}{2}\alpha_2 h^2 < \int_{h_0}^{h+h_0} F_2(\xi)\,\mathrm{d}\xi - F_2(h_0)\,h < \tfrac{1}{2}\beta_2 h^2, \tag{3.34b}$$

for all values of q_1, q_{10}, h and h_0. Substitution of (3.34) into (3.32) implies

$$\frac{1}{2}\iint_\Omega \{\nabla\eta\cdot\nabla\eta + \alpha_1(\Delta\eta+h)^2 - \beta_2 h^2\}\,\mathrm{d}x\,\mathrm{d}y$$
$$< \mathscr{L}(\boldsymbol{q}) < \frac{1}{2}\iint_\Omega \{\nabla\eta\cdot\nabla\eta + \beta_1(\Delta\eta+h)^2 - \alpha_2 h^2\}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.35}$$

We can now state the following nonlinear generalization of the formal stability result Theorem 3.2.

Theorem 3.4. *Suppose that the Casimir densities $F_1(\xi)$ and $F_2(\xi)$ which determine the steady solutions $h_0(x,y)$ and $\eta_0(x,y)$ through the relations (3.3) and (3.4) satisfy the convexity estimates for all ξ*

$$0 < \alpha_1 < F_1'(\xi) < \beta_1 < \infty, \tag{3.36}$$

$$-\infty < \alpha_2 < F_2'(\xi) < \beta_2 < 0, \tag{3.37}$$

for some real constants α_1, α_2, β_1 and β_2, then the steady solution $h_0(x,y)$ and $\eta_0(x,y)$ are nonlinearly stable in the sense of Liapunov with respect to the disturbance norm $\|\boldsymbol{q}\|$ given by

$$\|\boldsymbol{q}\|^2 = \iint_\Omega \nabla\eta\cdot\nabla\eta + (\Delta\eta)^2 + h^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.38}$$

Clearly, the conditions (3.36) and (3.37) are sufficient to establish the positive definiteness of $\mathscr{L}(\boldsymbol{q})$. All that remains to be established is the following estimate on the disturbance norm. It follows from (3.38) that

$$\begin{aligned}
\|\boldsymbol{q}\|^2 &= \iint_\Omega \nabla\eta\cdot\nabla\eta + (\Delta\eta + h - h)^2 + h^2\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \iint_\Omega \nabla\eta\cdot\nabla\eta + 2(\Delta\eta + h)^2 + 3h^2\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant 3\iint_\Omega \nabla\eta\cdot\nabla\eta + (\Delta\eta + h)^2 + h^2\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \tilde{\Gamma}\iint_\Omega \nabla\eta\cdot\nabla\eta + \alpha_1(\Delta\eta + h)^2 - \beta_2 h^2\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant 2\tilde{\Gamma}\mathscr{L}(\boldsymbol{q}),
\end{aligned} \tag{3.39a}$$

where the inequality $(x+y)^2 \leqslant 2(x^2+y^2)$ has been used and $\tilde{\Gamma}^{-1} \equiv \frac{1}{3}\min(1, \alpha_1, -\beta_2) > 0$ assuming (3.36) and (3.37). However, exploiting the invariance of $\mathscr{L}(\boldsymbol{q})$ and the inequality (3.35) we further find

$$\begin{aligned}
\|\boldsymbol{q}\|^2 &\leqslant 2\tilde{\Gamma}\mathscr{L}(\tilde{\boldsymbol{q}}) \\
&\leqslant \tilde{\Gamma}\iint_\Omega \nabla\tilde{\eta}\cdot\nabla\tilde{\eta} + \beta_1(\Delta\tilde{\eta} + \tilde{h})^2 - \alpha_2\tilde{h}^2\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \tilde{\Gamma}\iint_\Omega \nabla\tilde{\eta}\cdot\nabla\tilde{\eta} + 2\beta_1(\Delta\tilde{\eta})^2 + (2\beta_1 - \alpha_2)\,\tilde{h}^2\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \hat{\Gamma}\|\tilde{\boldsymbol{q}}\|^2,
\end{aligned} \tag{3.39b}$$

where $\tilde{\boldsymbol{q}} = \boldsymbol{q}(x,y,t=0)$ with corresponding definitions for $\tilde{\eta}(x,y)$ and $\tilde{h}(x,y)$, and $\hat{\Gamma} \equiv \Gamma\max(1, 2\beta_1, 2\beta_1 - \alpha_2) > 0$. It therefore follows from (3.39*b*) that

$$\|\boldsymbol{q}\| \leqslant (\hat{\Gamma})^{\frac{1}{2}}\,\|\tilde{\boldsymbol{q}}\|, \tag{3.39c}$$

which provides a rigorous nonlinear bound on the disturbance norm, assuming the conditions of Theorem 3.4 hold.

The nonlinear generalization of the formal stability result Theorem 3.2 can be obtained as follows. First, we need to introduce a Poincaré inequality of the form

$$\iint_{\Omega} \nabla\eta\cdot\nabla\eta \,\mathrm{d}x\,\mathrm{d}y \leqslant \tilde{C}\iint_{\Omega}(\Delta\eta)^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.40}$$

It is not immediately clear that a Poincaré inequality exists for every finite-amplitude disturbance $\eta(x,y,t)$. However, if we insist on *natural* boundary conditions for the disturbance fields, namely $\eta(\partial\Omega)\equiv 0$, then a Poincaré inequality of the form (3.40) will exist provided Ω is bounded in at least one direction. Assuming, then, that (3.40) holds, substitution into (3.35) leads to

$$\begin{aligned}\mathscr{L}(\boldsymbol{q}) &\leqslant \frac{1}{2}\iint_{\Omega}\tilde{C}(\Delta\eta)^2+\beta_1(\Delta\eta+h)^2-\alpha_2 h^2\,\mathrm{d}x\,\mathrm{d}y\\ &=\frac{1}{2}\iint_{\Omega}(\tilde{C}+\beta_1)(\Delta\eta+\gamma h)^2+(\tilde{C}\gamma-\alpha_2)h^2\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{3.41a}$$

where

$$\gamma\equiv\beta_1/(\tilde{C}+\beta_1). \tag{3.41b}$$

This expression is the generalization of (3.23) with the convexity constants (3.33). We have the following nonlinear generalization of the formal stability result Theorem 3.3.

Theorem 3.5. *Suppose that the Casimir densities $F_1(\xi)$ and $F_2(\xi)$ which determine the steady solutions $h_0(x,y)$ and $\eta_0(x,y)$ through the relations* (3.3) *and* (3.4) *satisfy the convexity estimates for all* ξ

$$-\infty<\alpha_1<F_1'(\xi)<\beta_1<-\tilde{C}<0, \tag{3.42}$$

$$\tilde{C}\gamma<\alpha_2<F_2'(\xi)<\beta_2<\infty, \tag{3.43}$$

for some real constants α_1, α_2, β_1 and β_2 where $\tilde{C}$ is the Poincaré constant given in (3.40) *associated with natural disturbances and γ is given by* (3.41*b*), *then the steady solutions $h_0(x,y)$ and $\eta_0(x,y)$ are nonlinearly stable in the sense of Liapunov with respect to the disturbance norm given by*

$$\|\boldsymbol{q}\|^2=\iint_{\Omega}(\Delta\eta)^2+h^2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.44}$$

Clearly, the conditions (3.42) and (3.43) are sufficient to establish the negative definiteness of $\mathscr{L}(\boldsymbol{q})$. All that needs to be established is the following estimate. It follows from (3.41), (3.42) and (3.43) that

$$\iint_{\Omega}(\Delta\eta+\gamma h)^2+h^2\,\mathrm{d}x\,\mathrm{d}y\leqslant 2\Gamma\mathscr{L}(\boldsymbol{q}), \tag{3.45}$$

where $\Gamma^{-1}\equiv\max(\tilde{C}+\beta_1,\tilde{C}\gamma-\alpha_2)<0$. On the other hand, it follows from (3.44) that

$$\begin{aligned}\|\boldsymbol{q}\|^2&=\iint_{\Omega}(\Delta\eta+\gamma h-\gamma h)^2+h^2\,\mathrm{d}x\,\mathrm{d}y\\&\leqslant\iint_{\Omega}2(\Delta\eta+\gamma h)^2+(1+2\gamma^2)h^2\,\mathrm{d}x\,\mathrm{d}y\\&\leqslant\tilde{\Gamma}\iint_{\Omega}(\Delta\eta+\gamma h)^2+h^2\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{3.46}$$

where $\tilde{\Gamma} \equiv \max(2, 1+2\gamma^2) > 0$ and where the inequality $(x+y)^2 \leqslant 2(x^2+y^2)$ has been used. It follows from (3.35), (3.45) and (3.46) that

$$\begin{aligned}\|\boldsymbol{q}\|^2 &\leqslant 2\tilde{\Gamma}\Gamma\mathscr{L}(\boldsymbol{q}) \equiv 2\tilde{\Gamma}\Gamma\mathscr{L}(\tilde{\boldsymbol{q}}) \\ &\leqslant \tilde{\Gamma}\Gamma \iint_\Omega \{\nabla\tilde{\eta}\cdot\nabla\tilde{\eta} + \alpha_1(\Delta\tilde{\eta}+\tilde{h})^2 - \beta_2\tilde{h}^2\}\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \tilde{\Gamma}\Gamma \iint_\Omega \nabla\tilde{\eta}\cdot\nabla\tilde{\eta} + 2\alpha_1(\Delta\tilde{\eta})^2 + (2\alpha_1-\beta_2)\,\tilde{h}^2\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \hat{\Gamma}\|\tilde{\boldsymbol{q}}\|^2, \end{aligned} \tag{3.46a}$$

where $\hat{\Gamma} \equiv \tilde{\Gamma}\Gamma \min(2\alpha_1, 2\alpha_1-\beta_2) > 0$, and where $\tilde{\boldsymbol{q}} = \boldsymbol{q}(x, y, t=0)$ with corresponding definitions for $\tilde{h}(x,y)$ and $\tilde{\eta}(x,y)$. It follows from (3.46a) that

$$\|\boldsymbol{q}\| \leqslant (\hat{\Gamma})^{\frac{1}{2}}\,\|\tilde{\boldsymbol{q}}\|, \tag{3.46b}$$

which provides a rigorous nonlinear bound on the disturbance norm, provided the conditions of Theorem 3.5 hold.

4. Isolated steadily-translating cold eddy solutions

(a) *Variational principles*

In this section we present two variational principles which can describe isolated steadily-travelling solutions to (2.10) and (2.11) and give a general discussion of the linear and nonlinear stability of these solutions. By isolated steadily-travelling solutions to (2.10) and (2.11) we mean sufficiently smooth solutions of the form

$$\eta = \eta_s(x-ct, y), \tag{4.1}$$

$$h = h_s(x-ct, y), \tag{4.2}$$

where c is the x-direction translation velocity such that $|\eta|$ and $|h|$ decay sufficient rapidly at infinity so that the globally integrated energy and enstrophy, given by, respectively,

$$\iint_\Omega \nabla\eta\cdot\nabla\eta + h^2\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{4.3a}$$

$$\iint_\Omega (\Delta\eta + h)^2\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{4.3b}$$

are finite. Note that solutions of the form (4.1) and (4.2) exclude the possibility of any y-direction translation. It is not difficult to show that (4.3a, b) exclude this possibility (see Flierl *et al.* (1980) for a similar discussion relating to the possible north–south motion of solitary planetary waves). It is important to point out that throughout this section it is assumed that either $\Omega = \mathbb{R}^2$ or the channel domain $\Omega = \{(x,y)\,|\,-\infty < x < \infty, |y| < y_{\mathrm{B}} < \infty\}$, and in either event we assume $\eta(\partial\Omega) = 0$ and $h(\partial\Omega) = 0$ if the support of h cannot be bounded away from the boundary of Ω given by $\partial\Omega$.

If (4.1) and (4.2) are substituted into (2.10) and (2.11), the result can be written in the form

$$\partial(\eta_s + cy, \Delta\eta_s + h_s - y) = 0, \tag{4.4}$$

$$\partial(\eta_s + (c-1)\,y, h_s) = 0, \tag{4.5}$$

where we write $\eta_s = \eta_s(\xi, y)$ and $h_s = h_s(\xi, y)$ with $\xi \equiv x-ct$, and $\partial(A,B) \equiv A_\xi B_y - A_y B_\xi$ and $\Delta \equiv \partial_{\xi\xi}+\partial_{yy}$. Equations (4.4) and (4.5) can be integrated to give

$$\eta_s + cy = \tilde{F}_1(\Delta\eta_s + h_s - y), \tag{4.6}$$

$$\eta_s + (c-1)\,y = \tilde{F}_2(h_s). \tag{4.7}$$

The relations (4.6) and (4.7) are just like (3.3) and (3.4) except that η_0 is replaced by the 'streaklines' $\eta_s + cy$, h_0 is replaced with h_s and the x-coordinate is replaced with the co-moving coordinate ξ.

Consider the functional $\mathscr{N}(\boldsymbol{q})$ given by

$$\mathscr{N}(\eta, h) = H(\boldsymbol{q}) + C(\boldsymbol{q}) - cM(\boldsymbol{q}), \tag{4.8}$$

where H, C and M are given by (2.20), (2.36) and (2.32), respectively, and where c will be the translation velocity in (4.1) and (4.2). Owing to the invariance of H, C and M, it follows $\mathscr{N}$ is an invariant of the full nonlinear dynamics (2.10) and (2.11). The first variation of $\mathscr{N}$ is given by

$$\delta\mathscr{N}(\eta, h) = \iint_\Omega \{(h-y)\,\delta h - (\eta + cy)\,\Delta\delta\eta + \Phi_1'(\Delta\eta + h - y)\,(\Delta\delta\eta + \delta h) + \Phi_2'(h)\,\delta h\}\,\mathrm{d}x\,\mathrm{d}y, \tag{4.9}$$

where the fact that $\eta(\partial\Omega) = 0$ has been used. Equation (4.9) can be rearranged into the form

$$\delta\mathscr{N}(\eta, h) = \iint_\Omega \{[\Phi_1'(q_1 - y) - \eta - cy]\,\delta q_1 + [\Phi_2'(q_2) + \eta + h + (c-1)\,y]\,\delta q_2\}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.10}$$

We can therefore state the variational principle.

Proposition 4.1. *Isolated steadily-travelling solutions of the form* $\eta = \eta_s(x-ct, y)$ *and* $h = h_s(x-ct, y)$ *as determined by* (4.6) *and* (4.7) *satisfy the first order necessary condition* $\delta\mathscr{N}(\eta_s, h_s) = 0$ *for extremizing the constrained hamiltonian* $\mathscr{N} = H + C - cM$ *provided the Casimir densities* $\Phi_1(q_1 - y)$ *and* $\Phi_2(q_2)$ *are given by*

$$\Phi_1(q_1 - y) \equiv \int_{-y}^{q_1 - y} \tilde{F}_1(\xi)\,\mathrm{d}\xi, \tag{4.11a}$$

$$\Phi_2(q_2) \equiv -\int_0^{q_2} \tilde{F}_2(\xi)\,\mathrm{d}\xi - \tfrac{1}{2}(q_2)^2. \tag{4.11b}$$

Swaters & Flierl (1991) found a class of exact radially-symmetric steadily-travelling isolated eddy solutions of the form

$$h_s(\xi, y) = \begin{cases} \tilde{h}_s(r), & 0 \leqslant r < a, \\ 0, & r \geqslant a, \end{cases} \tag{4.12a, b}$$

$$\eta_s(\xi, y) = \tilde{\eta}_s(r) = -\tfrac{1}{2}\pi Y_0(r)\int_0^r \xi J_0(\xi)\,\tilde{h}_s(\xi)\,\mathrm{d}\xi - \tfrac{1}{2}\pi J_0(r)\int_r^a \xi Y_0(\xi)\,\tilde{h}_s(\xi)\,\mathrm{d}\xi, \tag{4.13}$$

if $0 \leqslant r < a$, and

$$\eta_s(\xi, y) \equiv 0, \tag{4.14}$$

for $r \geqslant a$, where the translation velocity $c \equiv 1$, $r \equiv (\xi^2 + y^2)^{\frac{1}{2}}$, and where the *eddy radius* a must satisfy the 'isolation' condition

$$\int_0^a \xi J_0(\xi)\,\tilde{h}_s(\xi)\,\mathrm{d}\xi = 0, \tag{4.15}$$

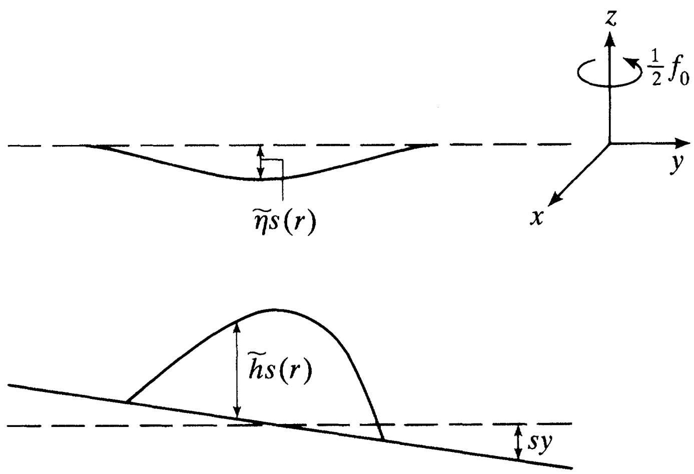

Figure 2. Geometry of a steadily-travelling isolated cold eddy such as given by (4.17).

where $J_0(r)$ and $Y_0(r)$ are the zero-order Bessel functions of the first and second kind, respectively. The constraint (4.15) can be physically interpreted as the necessary and sufficient condition for the annihilation of a topographic Rossby wave field in the region $r > a$ and concomitantly will imply that the relative circulation in layer one is zero and that $\tilde{\eta}_{s_r}(a) \equiv 0$.

For this class of isolated eddy solutions the functions $\tilde{F}_1(q_1)$ and $\tilde{F}_2(q_2)$ are given by, respectively,

$$\tilde{F}_1(q_1) = -q_1, \tag{4.16a}$$

$$\tilde{F}_2(q_2) = \tilde{\eta}_s(r_s(q_2)), \tag{4.16b}$$

where the function $r_s(q_2)$ is the inverse function associated with $q_2 = \tilde{h}_s(r)$ (in practical modelling situations it is not unreasonable to assume that $\tilde{h}_s(r)$ is monotonically decreasing with respect to r and hence one-to-one in the region where it is non-zero).

One particularly simple example of this class of solutions is given by

$$\tilde{h}_s(r) \equiv h_0[1-(r/a)^2], \tag{4.17a}$$

$$\tilde{\eta}_s(r) \equiv -\tilde{h}_s(r) + (rh_0/a^2)\,[J_0(r)/J_0(a)-1], \tag{4.17b}$$

where the eddy radius satisfies

$$J_2(a) = 0, \tag{4.17c}$$

where J_2 is the Bessel function of the first kind of order 2, and where the parameter h_0 is the maximum height of eddy located at $r \equiv 0$. For convenience we may rewrite (4.17*c*) as $a = j_{2,n}$ where $j_{2,n}$ is the nth non-trivial zero of J_2. The function $\tilde{F}_1(q_1)$ is given by (4.16*a*) and the function $\tilde{F}_2(q_2)$ in (4.16*b*) can be written in the form

$$\tilde{F}_2(q_2) = \tilde{\eta}_s(a[1-q_2/h_0]^{\frac{1}{2}}), \tag{4.18a}$$

or equivalently as

$$\tilde{F}_2(q_2) = -q_2 + (4h_0/a^2)\{J_0(a[1-q_2/h_0]^{\frac{1}{2}})/J_0(a)-1\}. \tag{4.18b}$$

Swaters & Flierl (1991) used this simple solution to explicitly calculate the trailing topographic Rossby wave field generated by a diabatically warming cold eddy translating on a sloping bottom in an attempt to model aspects of the evolution of a cold eddy observed on the New England Bight (Houghton *et al.* 1982).

Figure 2 shows the geometry associated with the solution (4.17). The cold eddy immediately above the sloping bottom has compact support and translates uniformly in the positive x-direction with the speed of unity. The cold eddy traps fluid particles and hence transports them along the sloping bottom. Immediately above the steadily-travelling eddy in layer one there exists a relatively intense low pressure region with corresponding cyclonic flow. Fluid particles in this region are not trapped however and are therefore not transported with the motion.

There is a second variational principle that can be found for isolated eddy solutions described by (4.4) and (4.5). Consider the functional $\tilde{\mathscr{N}}(\boldsymbol{q})$ given by

$$\tilde{\mathscr{N}}(\eta, h) \equiv \frac{1}{2}\iint_{\Omega} \nabla(\eta+cy)\cdot\nabla(\eta+cy)-c^2y^2+\chi_h[(h-y)^2-y^2]\,\mathrm{d}x\,\mathrm{d}y+C(\boldsymbol{q}), \quad (4.19)$$

where the second term in the integrand of the first integral is required to ensure that $|\tilde{\mathscr{N}}|$ is finite if Ω is unbounded. We can state the variational principle.

Corollary 4.2. *Isolated steady-travelling solutions of the form* $\eta = \eta_s(x-ct, y)$ *and* $h = h_s(x-ct, y)$ *as determined by* (4.6) *and* (4.7) *satisfy the first order necessary condition* $\delta\tilde{\mathscr{N}}(\eta_s, h_s) = 0$ *for extremizing the conserved functional* $\tilde{\mathscr{N}}$ *provided the Casimir densities* $\Phi_1(q_1-y)$ *and* $\Phi_2(q_2)$ *are given by* (4.11*a*, *b*) *respectively.*

Proof. We begin by showing that $\tilde{\mathscr{N}}$ is an invariant of the nonlinear dynamics. It follows from (4.19) that

$$\begin{aligned}\tilde{\mathscr{N}}_t &= \iint_{\Omega} -(\eta+cy)\,\Delta\eta_t+(h-y)\,h_t\,\mathrm{d}x\,\mathrm{d}y+C_t\\ &= H_t-cM_t+C_t \equiv 0, \qquad (4.20)\end{aligned}$$

since H, M and C are each invariant and where $\eta(\infty)\equiv 0$ has been used. The first variation of $\tilde{\mathscr{N}}$ is given by

$$\delta\tilde{\mathscr{N}} = \iint_{\Omega}\{-(\eta+cy)\,\Delta\delta\eta+(h-y)\,\delta h+\Phi_1'(\Delta\eta+h-y)\,(\Delta\delta\eta+\delta h)+\Phi_2'(h)\,\delta h\}\,\mathrm{d}x\,\mathrm{d}y,$$

where can be exactly arranged into the form

$$\delta\tilde{\mathscr{N}} = \delta\mathscr{N}, \quad (4.21)$$

where $\delta\mathscr{N}$ is given by (4.10). The Corollary is now an immediate consequence of Proposition 4.1. □

Benjamin (1984) pointed out that a functional similar in form to (4.19) derived in the context of barotropic quasi-geostrophic theory might be able to be used in an existence theory for solitary planetary waves. Much of the recent stability work on solitary planetary waves (see, for example, Laedke & Spatschek 1986; Petviashvili 1983; Swaters 1986; among others) can be interpreted as an attempt to establish the definiteness of the second variation of functionals very similar in construction to $\mathscr{N}$ and $\tilde{\mathscr{N}}$.

(*b*) *Implications of Andrews' Theorem*

In this section we show that there are no non-trivial steadily-travelling isolated eddy solutions which can satisfy the analogues of Theorems 3.2 and 3.3 for the functional $\mathscr{N}(\eta, h)$ given by (4.8). We begin our discussion by computing $\delta^2\mathscr{N}(\eta_s, h_s)$ from (4.10) which can be written in the form

$$\delta^2\mathscr{N}(\eta_s, h_s) \equiv \iint_{\Omega}\{\nabla(\delta\eta)\cdot\nabla(\delta\eta)+\Phi_{1s}''(\Delta\delta\eta+\delta h)^2+(\Phi_{2s}''+1)\,(\delta h)^2\}\,\mathrm{d}x\,\mathrm{d}y, \quad (4.22)$$

where $\Phi''_{1s} \equiv \mathrm{d}^2\Phi_1/\mathrm{d}(q_1-y)^2$ evaluated for $q_1-y = q_{1s} \equiv \Delta\eta_s + h_s - y$ and $\Phi''_{2s} \equiv \mathrm{d}^2\Phi_2/\mathrm{d}q_2^2$ evaluated for $q_2 = q_{2s} \equiv h_s$ where $\Phi_1(q_1-y)$ and $\Phi_2(q_2)$ are the Casimir densities given by (4.11a) and (4.11b), respectively, with $\tilde{F}_1$ and $\tilde{F}_2$ given by (4.6) and (4.7) respectively.

It is straightforward to show that $\delta^2\mathscr{N}(\eta_s, h_s)$ is an invariant of the linearized dynamics obtained by substituting $\eta = \eta_s(x-ct,y)+\delta\eta(x,y,t)$ and $h = h_s(x-ct,y)+\delta h(x,y,t)$ into (2.10) and (2.11) and neglecting all quadratic perturbation terms. The linear stability problem can be written in the form

$$(\partial_t + c\,\partial_x)(\Delta\delta\eta+\delta h)+\partial(\delta\eta-\Phi''_{1s}[\Delta\delta\eta+\delta\eta], \Delta\eta_s+h_s-y) = 0, \tag{4.23}$$

$$(\partial_t + c\,\partial_x)\,\delta h + \partial(\delta\eta+[\Phi''_{2s}+1]\,\delta h, h_s) = 0, \tag{4.24}$$

where (4.6), (4.7) and (4.11) have been used.

Note that (4.22) is exactly of the form (3.10) except, of course, that the coefficient functions Φ''_{10} and Φ''_{20} in (3.10) are replaced by Φ''_{1s} and Φ''_{2s}, respectively, in (4.22). Clearly, if there existed non-trivial steadily-travelling solutions $\eta_s(x-ct,y)$ and $h_s(x-ct,y)$ for which the Casimir densities Φ_{1s} and Φ_{2s} could satisfy the conditions of Theorems 3.2 or 3.3, where the appropriate space-time domain would be $\Omega\times[0,\infty)$ in place of Ω, then the linear stability in the sense of Liapunov of these steadily-translating solutions could be established. However, as we now show, there are no non-trivial isolated steadily-translating solutions which can satisfy these conditions. Our argument follows closely the approach taken by Andrews (1984) and Carnevale & Shepherd (1990).

If (4.6) and (4.7) are differentiated with respect to x, one obtains

$$\eta_{s_x} = \Phi''_{1s}(\Delta\eta_{s_x}+h_{s_x}), \tag{4.25a}$$

$$\eta_{s_x} = (\Phi''_{2s}+1)\,h_{s_x}. \tag{4.25b}$$

If (4.25a) is multiplied through by $(\Delta\eta_{s_x}+h_{s_x})$, (4.25$b$) can be used to obtain

$$\Phi''_{1s}(\Delta\eta_{s_x}+h_{s_x})^2+(\Phi''_{2s}+1)\,(h_{s_x})^2 = \eta_{s_x}\,\Delta\eta_{s_x}, \tag{4.26}$$

which when integrated over Ω can be put into the form

$$\iint_\Omega \nabla\eta_{s_x}\cdot\nabla\eta_{s_x}+\Phi''_{1s}(\Delta\eta_{s_x}+h_{s_x})^2+(\Phi''_{2s}+1)\,(h_{s_x})^2\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega}\eta_{s_x}\boldsymbol{n}\cdot\nabla\eta_{s_x}\,\mathrm{d}s. \tag{4.27}$$

In the case where $\Omega = \mathbb{R}^2$, the appropriate boundary conditions for the *isolated* eddy solutions that we are interested in is that $|\eta_s|\to 0$ *smoothly* as $x^2+y^2\to\infty$ for all $t\in[0,\infty)$. Hence we conclude the integral on the right-hand side of (4.27) is zero. For the channel domain $\Omega = \{(x,y)\,|\,-\infty<x<\infty,\ |y|<y_\mathrm{B}<\infty\}$, the appropriate boundary conditions on η_s is that it is constant on $y=\pm y_\mathrm{B}$ which implies $\eta_{s_x}\equiv 0$ on $y=\pm y_\mathrm{B}$ and again we conclude the integral on the right-hand side of (4.27) is zero. Hence we conclude that for all the situations we are interested in

$$\iint_\Omega \nabla\eta_{s_x}\cdot\nabla\eta_{s_x}+\Phi''_{1s}(\Delta\eta_{s_x}+h_{s_x})^2+(\Phi''_{2s}+1)\,(h_{s_x})^2\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{4.28}$$

Clearly, if $\Phi''_{1s}\geqslant 0$ and $\Phi''_{2s}+1\geqslant 0$ for all $(x,y,t)\in\Omega\times[0,\infty)$, which is the analogue of Theorem 3.2 for the steadily travelling solutions, then (4.28) implies $\eta_{s_x} = h_{s_x}\equiv 0$, which in turn implies that there are no non-trivial, isolated eddy solutions that can satisfy the conditions of Theorem 3.2.

The argument in relation to Theorem 3.3 proceeds in a similar fashion. It is straightforward to verify that η_{s_x} satisfies the Poincaré inequality

$$\iint_\Omega \nabla(\eta_{s_x})\cdot\nabla(\eta_{s_x})\,\mathrm{d}x\,\mathrm{d}y \leqslant \tilde{C}\iint_\Omega [\Delta(\eta_{s_x})]^2\,\mathrm{d}x\,\mathrm{d}y, \tag{4.29}$$

where the constant $\tilde{C}$ will be the same as the constant in (3.21) since $\eta_{s_x}(\partial\Omega)\equiv 0$ provided it is understood that we are only working with the channel domain $\Omega = \{(x,y)\,|\,-\infty < x < \infty,\ |y| < y_{\mathrm{B}} < \infty\}$. (In fact for this channel domain, the best constant is given by $\tilde{C} = (2y_{\mathrm{B}}/\pi)^2$.) Substitution of (4.29) into (4.28) implies

$$\begin{aligned} 0 &\leqslant \iint_\Omega \tilde{C}[\Delta(\eta_{s_x})]^2 + \Phi''_{1s}(\Delta\eta_{s_x} + h_{s_x})^2 + (\Phi''_{2s}+1)\,(h_{s_x})^2\,\mathrm{d}x\,\mathrm{d}y \\ &\equiv \iint_\Omega \{(\tilde{C}+\Phi''_{1s})\,[\Delta\eta_{s_x} + \tilde{\gamma}h_{s_x}]^2 + [\Phi''_{2s}+1+\tilde{C}\tilde{\gamma}]\,(h_{s_x})^2\}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4.30}$$

where $\tilde{\gamma} \equiv \Phi''_{1s}/(\tilde{C}+\Phi''_{1s})$.

Clearly, if $\Phi''_{1s} \leqslant -\tilde{C}$ and $\Phi''_{2s}+1 \leqslant -\tilde{C}\tilde{\gamma}$ for all $(x,y,t)\in\Omega\times[0,\infty)$, which is the analogue of Theorem 3.3 for the steadily-travelling solutions, then (4.30) implies $\eta_{s_x} = h_{s_x} \equiv 0$, which in turn implies that there are no non-trivial isolated eddy solutions that can satisfy the conditions of Theorem 3.3.

5. Summary

We have attempted in this paper to present a comprehensive linear and nonlinear stability analysis of the intermediate length scale model developed by Swaters & Flierl (1991) and Swaters (1991) for describing the dynamics of mesoscale gravity currents or coupled density fronts and steadily-translating isolated cold eddies on a sloping bottom. This new model has been successful in describing the principal dynamical features associated with diabatically warming cold eddies such as that observed on the New England Bight (Houghton *et al.* 1982) and the instability of coupled density fronts such as that observed by Griffiths *et al.* (1982).

It was shown that this model can be written as a non-canonical hamiltonian dynamical system. In addition, a Poisson bracket formalism was developed for the model. A basis for the kernel of the Poisson bracket was found, and these functionals, called Casimirs, could be used to develop a variational principle for describing arbitrary steady flows that are solutions of the model. Based on this variational principle, two sets of sufficient conditions (i.e. Theorems 3.2 and 3.3) were found that can establish the *formal* stability, and thus the linear stability in the sense of Liapunov, of these steady solutions. Assuming the conditions of Theorem 3.1 held, we were able to obtain an *a priori* bound on the disturbance energy within the context of the linear stability problem. As well, in the limit of a steady parallel shear flow aligned in the x-direction, it was shown that the conditions of Theorem 3.2 reduced to the normal-mode stability results obtained by Swaters (1991). It was also shown that, assuming the hypotheses of the second formal stability result given by Theorem 3.3 held, an *a priori* bound on the disturbance norm could be given in terms of the initial disturbance relative enstrophy and potential energy. Based on the formal stability analyses, explicit convexity estimates were found for the Casimir functionals that could provide suitable sufficient conditions that established the nonlinear stability in the sense of Liapunov for steady flow solutions. These convexity estimates are necessary to ensure that the steady solutions form a proper extremum of the constrained hamiltonian. Two nonlinear stability theorems are given as, respectively, the appropriate generalizations of the formal stability results. Based on these stability theorems, *nonlinear* bounds were derived for the disturbance norms.

Because the hamiltonian formalism is invariant under translations in the x-direction, the x-direction momentum (or impulse) functional is conserved. We were able to show that steadily-travelling isolated eddy solutions could satisfy the first-order necessary conditions for extremizing a suitably hamiltonian which included the x-direction momentum functional with a Lagrange multiplier given as the negative of the translation velocity. However, by exploiting a suitably extended form of Andrews' Theorems, we were able to show that it is not possible to use the energy-Casimir formalism developed in this paper to obtain conditions which could establish the stability of the isolated eddy solutions.

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